Алгоритмическая сложность неклассических логик унарного предиката

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НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ УНИВЕРСИТЕТ

Computational complexity of non-classical logics of an unary predicate

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  - the number and arity of predicate letters: any number of monadic decidable, a single binary undecidable.

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  NB This result can be strenghened to one monadic letter:
  - $R(x, y) \;\mapsto\; \Diamond(P(x) \wedge \Diamond P(y))$;
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- M. Rybakov, D. Shkatov 2018 **QInt**, as well as a number of related logics, including those containing the constant domain axiom, are undecidable in languages with two individual variables and a single monadic predicate letter.

This talk

Let L_{wfin} be the logic of finite (by the number of worlds) L-frames.
In this talk, we prove that:

- Every logic between $\mathbf{QK}_{wfin}$ and one of $\mathbf{QS5}_{wfin}$, $\mathbf{QGL.3}_{wfin}$, $\mathbf{QGrz.3}_{wfin}$ is not r.e. (Π^0_1-hard) in languages with three individual variables and an unrestricted supply of unary letters.
- Every logic between $\mathbf{QK}_{wfin}$ and one of $\mathbf{QKTB}_{wfin}$, $\mathbf{QGL}_{wfin}$, $\mathbf{QGrz}_{wfin}$ is not r.e. in languages with three individual variables and a single unary letters.
- (The positive fragment of) every logic between $\mathbf{QInt}_{wfin}$ and $\mathbf{QLC}_{wfin}$ is not r.e. in languages with three individual variables and an unrestricted supply of unary letters.
- (The positive fragment of) every logic between $\mathbf{QInt}_{wfin}$ and $\mathbf{QKC}_{wfin}$ is not r.e. in languages with three individual variables and a single unary letter.
- The same for the logics with the constant domain axiom.

NB D. Skvortsov 1995 $\mathbf{QInt}_{wfin}$ is not r.e.

Language

Intuitionostic formulas:

$$\varphi \quad := \quad P^n(x_1, \ldots, x_n) \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \to \varphi) \mid \forall x\ \varphi \mid \exists x\ \varphi$$

Modal formulas:

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We use the standard abbreviations:

$$\begin{array}{lcl} \neg\varphi & = & \varphi \to \bot; \\ \varphi \leftrightarrow \psi & = & (\varphi \to \psi) \wedge (\psi \to \varphi); \\ \Diamond\varphi & = & \neg\Box\neg\varphi. \end{array}$$

Kripke semantics

Kripke semantics

Kripke frame is a pair $\mathfrak{F} = \langle W, R \rangle$; for the intuitionistic language R is reflrxive, transitive, and antisymmetric.

Expanding domains. For a frame $\langle W, R \rangle$ consider a sysytem $(D_w)_{w \in W}$ of non-empty sets (domains) such that

$$(*) \quad wRw' \implies D_w \subseteq D_{w'}.$$

For every $w \in W$ define a classical model $\mathfrak{M}_w = (D_w, I_w)$.
For the intuitionistic case we aditionally claim:

$$wRw' \implies I_w(P^n) \subseteq I_{w'}(P^n).$$

This gives us a first-order Kripke model $\mathfrak{M} = (W, R, D, I)$ is a Kripke model, where $D = (D_w)_{w \in W}$ and $I = (I_w)_{w \in W}$.

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(Locally) constant domains. Replace $(*)$ with cd-condition:

$$(**) \quad wRw' \implies D_w = D_{w'}.$$

Predicate Kripke frames: an example

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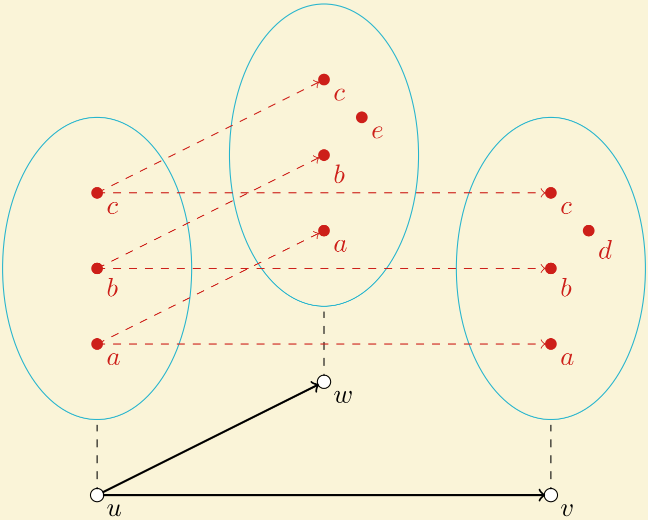

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Truth relation (intuitionistic language):

- $\mathfrak{M}, w \models^g P(x_1, \ldots, x_n)$ if $\langle g(x_1), \ldots, g(x_n) \rangle \in P^w$;
- $\mathfrak{M}, w \not\models^g \bot$;
- $\mathfrak{M}, w \models^g \varphi \wedge \psi$ if $\mathfrak{M}, w \models^g \varphi$ and $\mathfrak{M}, w \models^g \psi$;
- $\mathfrak{M}, w \models^g \varphi \vee \psi$ if $\mathfrak{M}, w \models^g \varphi$ or $\mathfrak{M}, w \models^g \psi$;
- $\mathfrak{M}, w \models^g \varphi \to \psi$ if $\mathfrak{M}, w' \models^g \varphi$ implies $\mathfrak{M}, w' \models^g \psi$, for any $w' \in R(w)$;
- $\mathfrak{M}, w \models^g \exists x\, \varphi$ if $\mathfrak{M}, w \models^{g'} \varphi$, for some g' s.t. $g' \stackrel{x}{=} g$ and $g'(x) \in D_w$;
- $\mathfrak{M}, w \models^g \forall x\, \varphi$ if $\mathfrak{M}, w' \models^{g'} \varphi$, for every $w' \in R(w)$ and every g' s.t. $g' \stackrel{x}{=} g$ and $g'(x) \in D_{w'}$.

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- $\mathfrak{M}, w \models^g \Box\varphi$ if $\mathfrak{M}, w' \models^g \varphi$, for every $w' \in R(w)$.

- $\mathfrak{M}, w \models \varphi(x_1, \ldots, x_n)$ if $\mathfrak{M}, w \models^g \varphi(x_1, \ldots, x_n)$, for every g such that $g(x_1), \ldots, g(x_n) \in D_w$;
- $\mathfrak{M} \models \varphi$ if $\mathfrak{M}, w \models \varphi$, for every $w \in W$;
- $\mathfrak{F} \models \varphi$ if $\mathfrak{M} \models \varphi$, for every model $\mathfrak{M}$ based over $\mathfrak{F}$.

Logics

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The logics under consideration are:

- $\mathbf{QCl}$, the classical predicate logic;
- $\mathbf{QCl}_{\mathit{fin}}$, the classical logic of finite models;
- $\mathbf{QK}$, the modal logic of all frames;
- $\mathbf{Q}L = \mathbf{QK} \oplus L$, for a normal modal propositional logic L;
- $\mathbf{QInt}$, the logic of all intuitionistic frames;
- $\mathbf{QLC}$, the logic of linear intuitionistic frames;
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It is known that $\mathbf{QCl}^{+\leqslant 2}_{\mathit{fin}}(3)$ is Π^0_1-complete.

Embedding of $\mathbf{QCl}^{+\leqslant 2}_{fin}(3)$ into $\mathbf{QK}_{wfin}$

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Let φ be a classical formula (in the language of $\mathbf{QCl}^{+\leqslant 2}_{fin}(3)$).
Let

$$\begin{array}{rcl} A_1 & = & \forall x\, \Diamond T(x); \\ A_2 & = & \forall x \forall y\, (x \approx y \leftrightarrow \Box (T(x) \leftrightarrow T(y))). \end{array}$$

Observe that A_2 implies that $\approx$ is an equivalence relation.
Let $A = A_1 \wedge A_2$ and let $Congr$ be the formula asserting that $\approx$ is a congruence with respect to the predicate letters of φ, i.e., a conjunction of formulas

$$\forall x \forall y\, (x \approx y \to (P(x) \to P(y)));$$
$$\forall x \forall y \forall z\, (x \approx y \to ((S(z,x) \to S(z,y)) \wedge (S(x,z) \to S(y,z))),$$

where P ranges over the monadic, and S binary, predicate letters of φ.
Lastly, let

$$\overline{\varphi} = A \wedge Congr \to \varphi.$$

Observe that $\overline{\varphi}$ contains three individual variables.

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Lemma

Let $L \in \{\mathbf{QK}, \mathbf{QS5}, \mathbf{QGL.3}, \mathbf{QGrz.3}\}$*. The following statements are equivalent:*

(1) $\varphi \in \mathbf{QCl}_{fin}$;

(2) $\overline{\varphi} \in L_{wfin}$;

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Theorem

Every logic in $[\mathbf{QK}_{wfin}, \mathbf{QGL.3.cd}_{wfin}]$, $[\mathbf{QK}_{wfin}, \mathbf{QGrz.3.cd}_{wfin}]$, *and* $[\mathbf{QK}_{wfin}, \mathbf{QS5}_{wfin}]$ *is* Π^0_1*-hard in languages with three individual variables and predicate letters of arity at most two.*

Eliminating of binary letters

Eliminating of binary letters

Let P be a binary predicate letters of $\overline{\varphi}$.
Let Q_1 and Q_2 be monadic predicate letters, not occurring in $\overline{\varphi}$.
Lastly, let $\cdot^\sigma$ be the function substituting $\Diamond(Q_1(x) \wedge Q_2(y))$ for $P(x, y)$.

Lemma

The following statements are equivalent:

(1) $\varphi \in \mathbf{QCl}_{\mathit{fin}}$;

(2) $\overline{\varphi}^\sigma \in \mathbf{QK}_{\mathit{wfin}}$;

(3) $\overline{\varphi}^\sigma \in \mathbf{QK.cd}_{\mathit{wfin}}$.

Proof.
(1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear.
We explain (3) $\Rightarrow$ (1) as $\neg$(1) $\Rightarrow$ $\neg$(3).

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Assume $\varphi \notin \mathbf{QCl}_{fin}$. Then, $\mu \not\Vdash \varphi$, for some classical model $\mu = \langle \mathcal{D}, \mathcal{I} \rangle$ with $\mathcal{D} = \{a_0, a_1, \ldots, a_n\}$. We define a $\mathbf{QK.cd}_{wfin}$-model $\mathfrak{M}$ and show that $\mathfrak{M}, w \not\models \overline{\varphi}$, for some $w \in W$.

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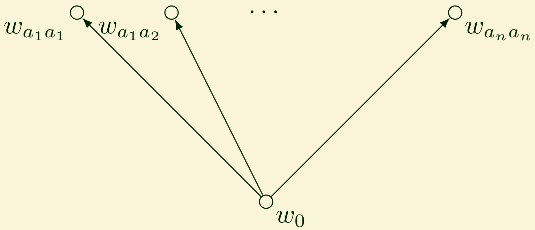
Idea:



We want: $w_{ab} \models Q_1(c) \wedge Q_2(d) \iff c = a, d = b, \mu \models P(a, b)$.

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Assume $\varphi \notin \mathbf{QCl}_{\mathit{fin}}$. Then, $\mu \not\Vdash \varphi$, for some classical model $\mu = \langle \mathcal{D}, \mathcal{I} \rangle$ with $\mathcal{D} = \{a_0, a_1, \ldots, a_n\}$. We define a $\mathbf{QK.cd}_{\mathit{wfin}}$-model $\mathfrak{M}$ and show that $\mathfrak{M}, w \not\models \overline{\varphi}$, for some $w \in W$. Let

- $W = \{w_0\} \cup \{w_{ab} : a, b \in \mathcal{D}\}$;
- $R = \{\langle w_0, w_{ab} \rangle : a, b \in \mathcal{D}\}$;
- $D_w = \mathcal{D}$, for every $w \in W$,

and let $I = (I_w)_{w \in W}$ be defined so that

- $w_{ab} \models T(c) \leftrightharpoons c = a$;
- $w_0 \models a_s \approx a_t \leftrightharpoons s = t$;
- $I_{w_0}(P) = \mathcal{I}(P)$, for every predicate letter P of φ;
- $w_{ab} \models Q_1(c) \leftrightharpoons c = a$;
- $w_{ab} \models Q_2(c) \leftrightharpoons c = b$ and $\mu \models P(a, b)$;

Finally, let $\mathfrak{M} = \langle W, R, D, I \rangle$.
Then, $w_0 \not\models \overline{\varphi}$. □

Single unary letter: modal language

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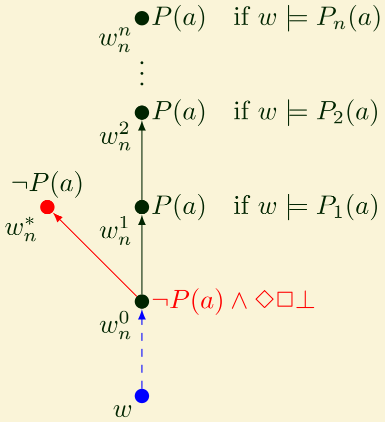
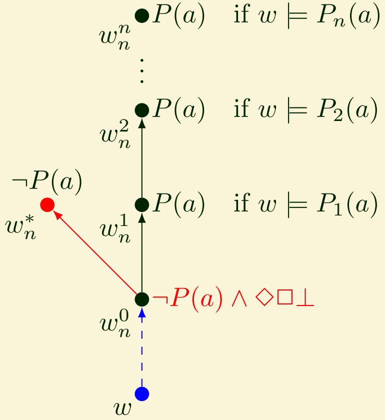


Single unary letter: modal language



Let $A_k(x) \;=\; \neg P(x) \wedge \Diamond\Box\bot \wedge \Diamond^n\Box\bot \wedge \Diamond^k P(x).$

Then the formula $\Diamond A_k(x)$ simulates $P_k(x)$ at the world w.

Single unary letter: modal language

Single unary letter: modal language

Theorem

Logics $\mathbf{QK}_{wfin}$ and $\mathbf{QK.cd}_{wfin}$ are Π^0_1-complete in the language with a single unary predicate letter and three individual variables.

Single unary letter: modal language

Theorem

Logics $\mathbf{QK}_{wfin}$ *and* $\mathbf{QK.cd}_{wfin}$ *are* Π^0_1*-complete in the language with a single unary predicate letter and three individual variables.*

Theorem

Logics $\mathbf{Q}L_{wfin}$ *and* $\mathbf{Q}L.\mathbf{cd}_{wfin}$ *are* Π^0_1*-hard in the language with a single unary predicate letter and three individual variables, for any* L *containing* $\mathbf{K}$ *and contained in one of* $\mathbf{GL}$, $\mathbf{Grz}$, $\mathbf{KTB}$.

Some results

Some results

- Let L be a logic containing $\mathbf{QK}$ and contained in $\mathbf{QGL} \oplus \boldsymbol{bf}$ or $\mathbf{QGrz} \oplus \boldsymbol{bf}$ or $\mathbf{QKTB} \oplus \boldsymbol{bf}$. Then L is Σ^0_1-hard in the language with a single unary predicate letter and two individual variables.

Some results

- Let L be a logic containing $\mathbf{QK}$ and contained in $\mathbf{QGL} \oplus \boldsymbol{bf}$ or $\mathbf{QGrz} \oplus \boldsymbol{bf}$ or $\mathbf{QKTB} \oplus \boldsymbol{bf}$. Then L is Σ^0_1-hard in the language with a single unary predicate letter and two individual variables.
- Let L be a logic containing $\mathbf{QK}$ and contained in $\mathbf{QS5}$. Then L is Σ^0_1-hard in the language with a two unary predicate letters, two individual variables, and infinitely many proposition letters.

Some results

- Let L be a logic containing **QK** and contained in $\mathbf{QGL} \oplus \boldsymbol{bf}$ or $\mathbf{QGrz} \oplus \boldsymbol{bf}$ or $\mathbf{QKTB} \oplus \boldsymbol{bf}$. Then L is Σ^0_1-hard in the language with a single unary predicate letter and two individual variables.
- Let L be a logic containing **QK** and contained in **QS5**. Then L is Σ^0_1-hard in the language with a two unary predicate letters, two individual variables, and infinitely many proposition letters.
- Let $\mathfrak{F} = \langle \mathbb{N}, R \rangle$, where R is a relation between $<$ and $\leqslant$. Then the logic of $\mathfrak{F}$ is Π^1_1-hard in the language with a single unary predicate letter, single proposition letter, and two individual variables.

Some results

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- The logic of finite frames of a logic contained in $\mathbf{QGL} \oplus \boldsymbol{bf}$, $\mathbf{QGrz} \oplus \boldsymbol{bf}$ or $\mathbf{QKTB} \oplus \boldsymbol{bf}$ is Π^0_1-hard in the language with a single unary predicate letter and three individual variables.

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- Let L be a logic containing $\mathbf{QwGrz}$ and contained in $\mathbf{QGL.3} \oplus \boldsymbol{bf}$ or $\mathbf{QGrz.3} \oplus \boldsymbol{bf}$. Then the logic of L-frames is Π^1_1-hard in the language with a single unary predicate letter, single proposition letter, and two individual variables.

Some results

- The logic of finite frames of a logic contained in $\mathbf{QGL} \oplus \boldsymbol{bf}$, $\mathbf{QGrz} \oplus \boldsymbol{bf}$ or $\mathbf{QKTB} \oplus \boldsymbol{bf}$ is Π^0_1-hard in the language with a single unary predicate letter and three individual variables.
- Let L be a logic containing $\mathbf{QwGrz}$ and contained in $\mathbf{QGL.3} \oplus \boldsymbol{bf}$ or $\mathbf{QGrz.3} \oplus \boldsymbol{bf}$. Then the logic of L-frames is Π^1_1-hard in the language with a single unary predicate letter, single proposition letter, and two individual variables.
- Predicate counterparts of $\mathbf{CTL}^*$, $\mathbf{CTL}$, $\mathbf{LTL}$, $\mathbf{ATL}^*$, $\mathbf{ATL}$ are Π^1_1-hard in the language with a single unary predicate letter and two individual variables.

Embedding of $\mathbf{QCl}_{fin}^{+\leqslant 2}(3)$ into $\mathbf{QInt}_{wfin}^{+}$

We construct an embedding of $\mathbf{QCl}_{fin}^{+\leqslant 2}(3)$ into positive fragment of any logic L between $\mathbf{QInt}_{wfin}$ and $\mathbf{QLC.cd}_{wfin}$.

Embedding of $\mathbf{QCl}^{+\leqslant 2}_{\mathit{fin}}(3)$ into $\mathbf{QInt}^{+}_{\mathit{wfin}}$

We construct an embedding of $\mathbf{QCl}^{+\leqslant 2}_{\mathit{fin}}(3)$ into positive fragment of any logic L between $\mathbf{QInt}_{\mathit{wfin}}$ and $\mathbf{QLC.cd}_{\mathit{wfin}}$.

Let

$$
\begin{array}{rcl}
Min(x) & = & \forall y\,(x \prec y \vee x \approx y);\\
Max(x) & = & \forall y\,(y \prec x);\\
x \lhd y & = & x \prec y \wedge \forall z\,(x \prec z \wedge z \prec y \to z \approx y).
\end{array}
$$

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Let

$$\begin{array}{rcl} Min(x) & = & \forall y\,(x \prec y \vee x \approx y); \\ Max(x) & = & \forall y\,(y \prec x); \\ x \lhd y & = & x \prec y \wedge \forall z\,(x \prec z \wedge z \prec y \to z \approx y). \end{array}$$

Define

$$\begin{array}{rcl} A_1 & = & \forall x \exists y\,(x \lhd y); \\ A_2 & = & \exists x\, Min(x); \\ A_3 & = & \exists x\, Max(x); \\ A_4 & = & \forall x \forall y \forall z\,(x \prec y \wedge y \prec z \to x \prec z); \\ A_5 & = & \forall x \forall y\,(x \prec y \vee y \prec x \vee x \approx y); \\ A_6 & = & \forall x \forall y\,(x \prec y \wedge y \prec x \to x \approx y); \\ A_7 & = & \forall x \forall y\,(x \prec y \wedge x \approx y \to Max(x)); \\ A_8 & = & \forall x \forall y\,((T(x) \to T(y)) \to x \prec y \vee x \approx y); \\ A_9 & = & \forall x \forall y\,(x \prec y \to (T(x) \to T(y))). \end{array}$$

Let A be a conjunction of formulas A_1 through A_9.

We construct an embedding of $\mathbf{QCl}^{+\leqslant 2}_{fin}(3)$ into positive fragment of any logic L between $\mathbf{QInt}_{wfin}$ and $\mathbf{QLC.cd}_{wfin}$.

Also let

$$B_1 = \forall x \forall y \forall z \bigwedge_{\psi \in sub(\varphi)} (q \to \psi);$$

$$B_2 = \forall x \forall y \forall z \bigwedge_{\psi \in sub(\varphi)} (\psi \vee (\psi \to q)),$$

and let $B = B_1 \wedge B_2$.

Embedding of $\mathbf{QCl}^{+\leqslant 2}_{fin}(3)$ into $\mathbf{QInt}^{+}_{wfin}$

We construct an embedding of $\mathbf{QCl}^{+\leqslant 2}_{fin}(3)$ into positive fragment of any logic L between $\mathbf{QInt}_{wfin}$ and $\mathbf{QLC.cd}_{wfin}$.

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and let $B = B_1 \wedge B_2$.

Let C be a conjunction of the formulas

$$\forall x\,(x \approx x) \wedge \forall x \forall y\,(x \approx y \to y \approx x) \wedge \forall x \forall y \forall z\,(x \approx y \wedge y \approx z \to x \approx z);$$
$$\forall x \forall y\,\big(x \approx y \to (P(x) \to P(y))\big);$$
$$\forall x \forall y \forall z\,\big(x \approx y \to ((S(z,x) \to S(z,y)) \wedge (S(x,z) \to S(y,z))\big),$$

where P ranges over the monadic, and S binary, predicate letters of φ.

Embedding of $\mathbf{QCl}^{+\leqslant 2}_{fin}(3)$ into $\mathbf{QInt}^{+}_{wfin}$

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$$\overline{\varphi} \quad = \quad A \wedge B \wedge C \to \varphi.$$

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Lemma

Let $L \in \{\mathbf{QInt}, \mathbf{QLC}\}$. The following statements are equivalent:

(1) $\varphi \in \mathbf{QCl}_{\mathit{fin}}$;

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Proof.

We construct an embedding of $\mathbf{QCl}^{+\leqslant 2}_{fin}(3)$ into positive fragment of any logic L between $\mathbf{QInt}_{wfin}$ and $\mathbf{QLC.cd}_{wfin}$.

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Proof.
(2) $\Rightarrow$ (3): obvious.

Embedding of $\mathbf{QCl}^{+\leqslant 2}_{fin}(3)$ into $\mathbf{QInt}^{+}_{wfin}$

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Proof.
(2) $\Rightarrow$ (3): obvious.
(1) $\Rightarrow$ (2): technical.

Embedding of $\mathbf{QCl}^{+\leqslant 2}_{fin}(3)$ into $\mathbf{QInt}^{+}_{wfin}$

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(3) $\overline{\varphi} \in L.\mathbf{cd}_{wfin}$.

Proof.

(2) $\Rightarrow$ (3): obvious.

(1) $\Rightarrow$ (2): technical.

(3) $\Rightarrow$ (1): we need it; we prove $\neg(1) \Rightarrow \neg(3)$.

Embedding of $\mathbf{QCl}^{+\leqslant 2}_{fin}(3)$ into $\mathbf{QInt}^{+}_{wfin}$

$\neg(1) \Rightarrow \neg(3)$:

Embedding of $\mathbf{QCl}^{+\leqslant 2}_{fin}(3)$ into $\mathbf{QInt}^{+}_{wfin}$

$\neg(1) \Rightarrow \neg(3)$:

Assume $\varphi \notin \mathbf{QCl}_{fin}$. Then, $\mu \not\Vdash \varphi$, for some classical model $\mu = \langle \mathcal{D}, \mathcal{I} \rangle$ with $\mathcal{D} = \{a_0, a_1, \ldots, a_n\}$. We define a $\mathbf{QLC.cd}_{wfin}$-model $\mathfrak{M}$ and show that $\mathfrak{M}, w \not\models \overline{\varphi}$, for some $w \in W$. Let

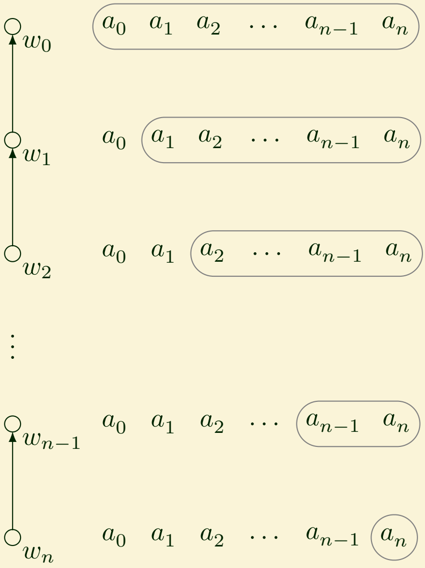
- $W = \{w_0, w_1, \ldots, w_n\}$;
- $R = \{\langle w_k, w_{k-1} \rangle : 1 \leqslant k \leqslant n\}^*$;
- $D_w = \mathcal{D}$, for every $w \in W$,

and let $I = (I_w)_{w \in W}$ be defined so that

- $w_k \models T(a_s) \leftrightharpoons k \leqslant s$;
- $w_k \models a_s \prec a_t \leftrightharpoons$ either $s < t$ or both $s \geqslant k$ and $t \geqslant k$;
- $w_k \models a_s \approx a_t \leftrightharpoons$ either $s = t$ or both $s \geqslant k$ and $t \geqslant k$;
- $w_k \models q \leftrightharpoons k \neq n$;
- $I_{w_n}(P) = \mathcal{I}(P)$, for every predicate letter P of φ;
- $I_{w_k}(P) = \mathcal{D}^m$, for every $k \neq n$ and m-ary predicate letter P of φ.

Finally, let $\mathfrak{M} = \langle W, R, D, I \rangle$. Evidently, I satisfies the heredity condition; therefore, $\mathfrak{M}$ is a $\mathbf{QLC.cd}_{wfin}$-model.
Then, $w_n \not\models \overline{\varphi}$. □

Embedding of $\mathbf{QCl}^{+\leqslant 2}_{fin}(3)$ into $\mathbf{QInt}^{+}_{wfin}$



Embedding of $\mathbf{QCl}^{+\leqslant 2}_{fin}(3)$ into $\mathbf{QInt}^{+}_{wfin}$

Theorem

Every logic between $\mathbf{QInt}_{wfin}$ and $\mathbf{QLC.cd}_{wfin}$ is Π^0_1-hard and Σ^0_1-hard in languages with three individual variables and predicate letters of arity at most two.

Embedding of $\mathbf{QCl}^{+\leqslant 2}_{\mathit{fin}}(3)$ into $\mathbf{QInt}^{+}_{\mathit{wfin}}$

Theorem

Every logic between $\mathbf{QInt}_{\mathit{wfin}}$ and $\mathbf{QLC.cd}_{\mathit{wfin}}$ is Π^0_1-hard and Σ^0_1-hard in languages with three individual variables and predicate letters of arity at most two.

Thus, many predicate superintuitionistic logics of natural classes of finite Kripke frames are neither recursively enumerable nor co-recursively enumerable in such languages:

Corollary

Let $L \in \{\mathbf{QInt}, \mathbf{QKP}, \mathbf{QLM}, \mathbf{QKC}, \mathbf{QLC}\}$.
Then, L_{wfin} and $L.\mathbf{cd}_{\mathit{wfin}}$ are both Π^0_1-hard and Σ^0_1-hard in languages with three individual variables and predicate letters of arity at most two.

Elimination of binary predicate letters

Elimination of binary predicate letters

Let $P_1, \dots, P_m$ be the binary predicate letters of $\overline{\varphi}$.

Let $F_1, G_1, \dots, F_m, G_m$ be distinct monadic predicate letters, and $p_1, r_1, \dots, p_m, r_m$ distinct proposition letters, not occurring in $\overline{\varphi}$.

Lastly, let $\cdot^\sigma$ be the function substituting $(F_j(x) \wedge G_j(y) \to p_j) \vee r_j$ for $P_j(x, y)$, for each $j \in \{1, \dots, m\}$, in $\overline{\varphi}$.

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Lemma

Let $L \in \{\mathbf{QInt}, \mathbf{QKC}\}$. The following statements are equivalent:

(1) $\varphi \in \mathbf{QCl}_{\mathit{fin}}$;

(2) $\overline{\varphi}^\sigma \in L_{\mathit{wfin}}$;

(3) $\overline{\varphi}^\sigma \in L.\mathbf{cd}_{\mathit{wfin}}$.

Proof. Similar to Kripke trick. □

Elimination of binary predicate letters

Elimination of binary predicate letters

Let $q_1, \dots, q_m$ be the proposition letters of $\overline{\varphi}^\sigma$ and let $Q_1, \dots, Q_m$ be distinct monadic predicate letters not occurring in $\overline{\varphi}^\sigma$. Let $\overline{\varphi}^\#$ be the result of substituting $\exists x\, Q_i(x)$ for q_i, for each $i \in \{1, \dots, m\}$, in $\overline{\varphi}^\sigma$.

Corollary

Let $L \in \{\mathbf{QInt}, \mathbf{QKC}\}$. The following statements are equivalent:

(1) $\varphi \in \mathbf{QCl}_{\mathit{fin}}$;

(2) $\overline{\varphi}^\# \in L_{\mathit{wfin}}$;

(3) $\overline{\varphi}^\# \in L.\mathbf{cd}_{\mathit{wfin}}$.

We, therefore, obtain the following:

Theorem

Every logic between $\mathbf{QInt}_{\mathit{wfin}}$ and $\mathbf{QKC.cd}_{\mathit{wfin}}$ is Π^0_1-hard in languages with three individual variables and only monadic predicate letters.

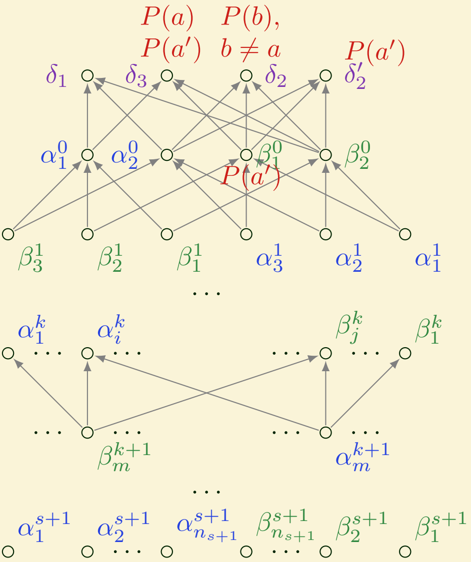
Elimination of monadic predicate letters

Elimination of monadic predicate letters

Let $P_1, \ldots, P_s$ be the (monadic) predicate letters of $\overline{\varphi}^{\#}$. We assume that $s \geqslant 2$—otherwise, $\overline{\varphi}^{\#}$ already has the required form. Let P be a monadic predicate letter distinct from $P_1, \ldots, P_s$.

We begin by defining a finite predicate frame $\mathfrak{F}_0 = \langle W_0, R_0 \rangle$ and some special model with cd-condition on it defined for an individual a; we assume that the domain of the model contains at least three element; we refer to such a model as a-suitable.

Elimination of monadic predicate letters



Elimination of monadic predicate letters

First, we define formulas associated with the worlds of the three top-most levels:

$$
\begin{array}{lcl}
D_1 & = & \exists x\, P(x); \\
D_2(x) & = & \exists x\, P(x) \to P(x); \\
D_3(x) & = & P(x) \to \forall x\, P(x); \\
\\
A_1^0(x) & = & D_2(x) \to D_1 \vee D_3(x); \\
A_2^0(x) & = & D_3(x) \to D_1 \vee D_2(x); \\
B_1^0(x) & = & D_1 \to D_2(x) \vee D_3(x); \\
B_2^0(x) & = & A_1^0(x) \wedge A_2^0(x) \wedge B_1^0(x) \to D_1 \vee D_2(x) \vee D_3(x); \\
\\
A_1^1(x) & = & A_1^0(x) \wedge A_2^0(x) \to B_1^0(x) \vee B_2^0(x); \\
A_2^1(x) & = & A_1^0(x) \wedge B_1^0(x) \to A_2^0(x) \vee B_2^0(x); \\
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B_3^1(x) & = & B_1^0(x) \wedge B_2^0(x) \to A_1^0(x) \vee A_2^0(x).
\end{array}
$$

Elimination of monadic predicate letters

We proceed by recursion. Assume formulas associated with the worlds of level k, where $k \geqslant 1$, have been defined. Let i, j and m be as in the definition of frame $\mathfrak{F}_0$ above; put

$$\begin{array}{rcl} A_m^{k+1}(x) & = & A_1^k(x) \to B_1^k(x) \vee A_i^k(x) \vee B_j^k(x); \\ B_m^{k+1}(x) & = & B_1^k(x) \to A_1^k(x) \vee A_i^k(x) \vee B_j^k(x). \end{array}$$

Elimination of monadic predicate letters

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Lemma

Let $\mathfrak{N}_a$ be an a-suitable model with a constant domain $\mathcal{A}$. Then,

$$\begin{array}{lcl} \mathfrak{N}_a, w \not\models A^k_m(a) & \iff & wR_0\alpha^k_m; \\ \mathfrak{N}_a, w \not\models B^k_m(a) & \iff & wR_0\beta^k_m. \end{array}$$

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Lemma

Let $\mathfrak{N}_a$ be an a-suitable model with a constant domain $\mathcal{A}$ and let $b \in \mathcal{A} - \{a\}$. Then, for every $w \in W_0$ and every $k \geqslant 2$,

$$\mathfrak{N}_a, w \models A^k_m(b) \quad \textit{and} \quad \mathfrak{N}_a, w \models B^k_m(b).$$

Elimination of monadic predicate letters

Elimination of monadic predicate letters

Let $(\overline{\varphi}^{\#})'$ be the result of substituting into $\overline{\varphi}^{\#}$, for each $r \in \{1, \dots, s\}$,

$$A_r^{s+1}(x) \vee B_r^{s+1}(x) \quad \text{for} \quad P_r(x).$$

Lemma

Let $L \in \{\mathbf{QInt}, \mathbf{QKC}\}$. The following statements are equivalent:

(1) $\varphi \in \mathbf{QCl}_{fin}$;

(2) $(\overline{\varphi}^{\#})' \in L_{wfin}$;

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Proof.

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Proof.

(1) ⇒ (2) ⇒ (3): obvious.

Elimination of monadic predicate letters

Let $(\overline{\varphi}^{\#})'$ be the result of substituting into $\overline{\varphi}^{\#}$, for each $r \in \{1, \ldots, s\}$,

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Let $L \in \{\mathbf{QInt}, \mathbf{QKC}\}$*. The following statements are equivalent:*

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Proof.

(1) $\Rightarrow$ (2) $\Rightarrow$ (3): obvious.

(3) $\Rightarrow$ (1): we prove it as $\neg(1) \Rightarrow \neg(3)$.

Elimination of monadic predicate letters

Case $\mathbf{QInt.cd}_{wfin}$:

Assume $\varphi \notin \mathbf{QCl}_{fin}$. Then $\mathfrak{M}^{\#}, w_n \not\models \overline{\varphi}^{\#}$, where $\mathfrak{M}^{\#}$ is the model constructed on finite linear model $\mathfrak{M}$ with cd-condition: the domain of any its world is $\mathcal{D} = \{a_0, a_1, \ldots, a_n\}$; we may assume that $\mathcal{D}$ contains at least three elements.

We use $\mathfrak{M}^{\#}$ to obtain a finite intuitionistic Kripke model with a constant domain refuting $(\overline{\varphi}^{\#})'$.

For every $a \in \mathcal{D}$, let $\mathfrak{F}^a = \langle \{a\} \times W_0, R^a \rangle$ be an isomorphic copy of the frame $\mathfrak{F}_0$ under the isomorphism $f\colon v \mapsto \langle a, v \rangle$.

Let

$$W'' \quad = \quad W' \cup (\mathcal{D} \times W_0).$$

Since W', $\mathcal{D}$ and W_0 are finite, so is W''.

Elimination of monadic predicate letters

Let S be the smallest relation on W'' such that

- $R' \subseteq S$;
- $\bigcup\limits_{a \in \mathcal{D}} R^a \subseteq S$;
- for every $w \in W'$, every $v \in W'' - W'$, every $a \in \mathcal{D}$ and every $r \in \{1, \ldots, s\}$,

$$wSv \quad \leftrightharpoons \quad \begin{array}{ll} \text{either} & v \in \{\langle a, \alpha_r^{s+1}\rangle, \langle a, \beta_r^{s+1}\rangle\} \text{ and } \mathfrak{M}', w \not\models P_r(a) \\ \text{or} & v \in \{\langle a, \alpha_{s+1}^{s+1}\rangle, \langle a, \beta_{s+1}^{s+1}\rangle\}, \end{array}$$

and let R'' be the reflexive transitive closure of S.

Let $D''(u) = \mathcal{D}$, for every $u \in W''$.

Elimination of monadic predicate letters

Let I'' be an interpretation on $\langle W'', R'', D''\rangle$ such that, for every $a \in \mathcal{D}$,

- $I''_{\langle a,\delta_2\rangle}(P) = \mathcal{D} - \{a\}$;
- $I''_{\langle a,\delta'_2\rangle}(P) = \{a'\}$, where $a' \equiv (a+1) \mod |\mathcal{D}|$;
- $I''_{\langle a,\delta_3\rangle}(P) = \{a, a'\}$, where $a' \equiv (a+1) \mod |\mathcal{D}|$;
- $I''_{\langle a,\beta_1^0\rangle}(P) = \{a'\}$, where $a' \equiv (a+1) \mod |\mathcal{D}|$;
- $I''_u(P) = \varnothing$, for $u \in W'' - \{\langle c,\delta_2\rangle, \langle c,\delta'_2\rangle, \langle c,\delta_3\rangle, \langle c,\beta_1^0\rangle : c \in \mathcal{D}\}$.

Finally, let $\mathfrak{M}'' = \langle W'', R'', D'', I''\rangle$.

Evidently, I'' satisfies the heredity condition; hence, $\mathfrak{M}''$ is an intuitionistic Kripke model.

Observe that, for any $a \in \mathcal{D}$, the submodel of $\mathfrak{M}''$ generated by the set

$$\{\langle a,\alpha_1^{s+1}\rangle, \ldots, \langle a,\alpha_{n_{s+1}}^{s+1}\rangle, \langle a,\beta_1^{s+1}\rangle, \ldots, \langle a,\beta_{n_{s+1}}^{s+1}\rangle\}$$

is an a-suitable model based on a frame isomorphic, under the isomorphism $f\colon v \mapsto \langle a, v\rangle$, to $\mathfrak{F}_0$.

Elimination of monadic predicate letters

Sublemma

For every $w \in W'$ and $a \in \mathcal{D}$,

$$\mathfrak{M}'', w \not\models A_1^s(a) \quad \text{and} \quad \mathfrak{M}'', w \not\models B_1^s(a).$$

Elimination of monadic predicate letters

Sublemma

For every $w \in W'$ and $a \in \mathcal{D}$,

$$\mathfrak{M}'', w \not\models A_1^s(a) \quad \text{and} \quad \mathfrak{M}'', w \not\models B_1^s(a).$$

Sublemma

$\mathfrak{M}'', v \models^g \psi'$, for every $\psi \in sub(\overline{\varphi}^{\#})$, every $v \in W'' - W'$ and every assignment g.

Elimination of monadic predicate letters

Sublemma

For every $w \in W'$ and $a \in \mathcal{D}$,

$$\mathfrak{M}'', w \not\models A_1^s(a) \quad \text{and} \quad \mathfrak{M}'', w \not\models B_1^s(a).$$

Sublemma

$\mathfrak{M}'', v \models^g \psi'$, for every $\psi \in sub(\overline{\varphi}^{\#})$, every $v \in W'' - W'$ and every assignment g.

We now show that $\mathfrak{M}'', w_n \not\models (\overline{\varphi}^{\#})'$.

To that end, we prove that, for every $w \in W'$, every $\theta \in sub(\overline{\varphi}^{\#})$ and every assignment g,

$$\mathfrak{M}^{\#}, w \models^g \theta \quad \iff \quad \mathfrak{M}'', w \models^g \theta'.$$

Elimination of monadic predicate letters

Let $\theta = P_r(x)$, and so $\theta' = A_r^{s+1}(x) \vee B_r^{s+1}(x)$, for some $r \in \{1, \ldots, s\}$.

Assume $\mathfrak{M}^{\#}, w \not\models P_r(a)$. By definition of $\mathfrak{M}''$, both $wR''\langle a, \alpha_r^{s+1}\rangle$ and $wR''\langle a, \beta_r^{s+1}\rangle$.

Then, both $\mathfrak{M}'', \langle a, \alpha_r^{s+1}\rangle \not\models A_r^{s+1}(a)$ and $\mathfrak{M}'', \langle a, \beta_r^{s+1}\rangle \not\models B_r^{s+1}(a)$.

Hence, by heredity, $\mathfrak{M}'', w \not\models A_r^{s+1}(a)$ and $\mathfrak{M}'', w \not\models B_r^{s+1}(a)$.

Therefore, $\mathfrak{M}'', w \not\models A_r^{s+1}(a) \vee B_r^{s+1}(a)$.

Elimination of monadic predicate letters

Conversely, assume $\mathfrak{M}'', w \not\models A_r^{s+1}(a) \vee B_r^{s+1}(a)$. Then, $\mathfrak{M}'', w \not\models A_r^{s+1}(a)$ and $\mathfrak{M}'', w \not\models B_r^{s+1}(a)$. Hence, there exist $u', u'' \in W''$ and i, j (corresponding to r) such that $u', u'' \in w{\uparrow}$ and

$$u' \models A_1^s(a); \quad u' \not\models B_1^s(a); \quad u' \not\models A_i^s(a); \quad u' \not\models B_j^s(a);$$
$$u'' \models B_1^s(a); \quad u'' \not\models A_1^s(a); \quad u'' \not\models A_i^s(a); \quad u'' \not\models B_j^s(a).$$

We show that $u' = \langle a, \alpha_r^{s+1} \rangle$ and $u'' = \langle a, \beta_r^{s+1} \rangle$.
Since $u' \models A_1^s(a)$ and $u'' \models B_1^s(a)$, by the first sublemma, $u', u'' \in W'' - W'$. Therefore, from $u' \not\models B_1^s(a)$ and $u'' \not\models A_1^s(a)$ we obtain that $u', u'' \in \{a\} \times W_0$. Hence,

$$\neg u' R'' \langle a, \alpha_1^s \rangle; \quad u' R'' \langle a, \beta_1^s \rangle; \quad u' R'' \langle a, \alpha_i^s \rangle; \quad u' R'' \langle a, \beta_j^s \rangle;$$
$$\neg u'' R'' \langle a, \beta_1^s \rangle; \quad u'' R'' \langle a, \alpha_1^s \rangle; \quad u'' R'' \langle a, \alpha_i^s \rangle; \quad u'' R'' \langle a, \beta_j^s \rangle.$$

Now, in $\mathfrak{F}_0$, and hence in $\mathfrak{F}^a$, only worlds of level $s+1$ see more than one world of level s. Therefore, u' and u'' are worlds of level $s+1$. Then, $u' = \langle a, \alpha_r^{s+1} \rangle$, $u'' = \langle a, \beta_r^{s+1} \rangle$, and $wR'' \langle a, \alpha_r^{s+1} \rangle$, $wR'' \langle a, \beta_r^{s+1} \rangle$. Hence, by definition of R'', we obtain $\mathfrak{M}^{\#}, w \not\models P_r(a)$.

Elimination of monadic predicate letters

The cases $\theta = \psi \vee \chi$, $\theta = \psi \wedge \chi$ and $\theta = \exists x\, \psi$ are straightforward.

Assume $\mathfrak{M}^{\#}, w \not\models^g \psi \to \chi$. Then, $\mathfrak{M}^{\#}, v \models^g \psi$ and $\mathfrak{M}^{\#}, v \not\models^g \chi$, for some v such that $wR'v$ (and so $wR''v$).
By inductive hypothesis, $\mathfrak{M}'', v \models^g \psi'$ and $\mathfrak{M}'', v \not\models^g \chi'$.
Therefore, $\mathfrak{M}'', w \not\models^g \psi' \to \chi'$.

Conversely, assume $\mathfrak{M}'', w \not\models^g \psi' \to \chi'$. Then, $\mathfrak{M}'', v \models^g \psi'$ and $\mathfrak{M}'', v \not\models^g \chi'$, for some v such that $wR''v$.
By the second sublemma, $v \in W'$, and so $wR'v$.
Hence, by inductive hypothesis, $\mathfrak{M}^{\#}, v \models^g \psi$ and $\mathfrak{M}^{\#}, v \not\models^g \chi$.
Therefore, $\mathfrak{M}^{\#}, w \not\models^g \psi \to \chi$.

Elimination of monadic predicate letters

Assume $\mathfrak{M}^{\#}, w \not\models^{g} \forall x\, \psi$.
Then, $\mathfrak{M}^{\#}, v \not\models^{g'} \psi$, for some v such that $wR'v$ (and so $wR''v$) and some g' such that $g' \stackrel{x}{=} g$.
By inductive hypothesis, $\mathfrak{M}'', v \models^{g'} \psi'$. Therefore, $\mathfrak{M}, w \not\models^{g} \forall x\, \psi'$.

Conversely, assume $\mathfrak{M}'', w \not\models^{g} \forall x\, \psi'$.
Then, $\mathfrak{M}'', v \not\models^{g'} \psi'$, for some v such that $wR''v$ and some g' such that $g' \stackrel{x}{=} g$.
By the second sublemma, $v \in W$, and so $wR'v$.
Hence, by inductive hypothesis, $\mathfrak{M}^{\#}, v \not\models^{g'} \psi$.
Therefore, $\mathfrak{M}^{\#}, w \not\models^{g} \forall x\, \psi$.

This completes the induction.

Since $w_n \in W'$, it follows from the claim proven by induction that $\mathfrak{M}'', w_n \not\models (\overline{\varphi}^{\#})'$. Therefore, $(\overline{\varphi}^{\#})' \notin \mathbf{QInt.cd}_{wfin}$.

Elimination of monadic predicate letters

Assume $\mathfrak{M}^{\#}, w \not\models^{g} \forall x\, \psi$.
Then, $\mathfrak{M}^{\#}, v \not\models^{g'} \psi$, for some v such that $wR'v$ (and so $wR''v$) and some g' such that $g' \stackrel{x}{=} g$.
By inductive hypothesis, $\mathfrak{M}'', v \models^{g'} \psi'$. Therefore, $\mathfrak{M}, w \not\models^{g} \forall x\, \psi'$.

Conversely, assume $\mathfrak{M}'', w \not\models^{g} \forall x\, \psi'$.
Then, $\mathfrak{M}'', v \not\models^{g'} \psi'$, for some v such that $wR''v$ and some g' such that $g' \stackrel{x}{=} g$.
By the second sublemma, $v \in W$, and so $wR'v$.
Hence, by inductive hypothesis, $\mathfrak{M}^{\#}, v \not\models^{g'} \psi$.
Therefore, $\mathfrak{M}^{\#}, w \not\models^{g} \forall x\, \psi$.

This completes the induction.

Since $w_n \in W'$, it follows from the claim proven by induction that $\mathfrak{M}'', w_n \not\models (\overline{\varphi}^{\#})'$. Therefore, $(\overline{\varphi}^{\#})' \notin \mathbf{QInt.cd}_{wfin}$.

Case $\mathbf{QKC.cd}_{wfin}$: just add a top point to $\mathfrak{M}''$. $\square$

Elimination of monadic predicate letters

We, thus, obtain the following:

Theorem

Every logic between $\mathbf{QInt}_{wfin}$ *and* $\mathbf{QKC.cd}_{wfin}$ *is* Π^0_1*-hard in languages with three individual variables and a single monadic predicate letter.*

Elimination of monadic predicate letters

In particular, we obtain the following:

Corollary

Let $L \in \{\mathbf{QInt}, \mathbf{QKP}, \mathbf{QLM}, \mathbf{QKC}\}$. Then, L_{wfin} and $L.\mathbf{cd}_{wfin}$ are Π^0_1-hard in languages with three individual variables and a single monadic predicate letter.

Since every consistent propositional superintuitionistic logic distinct from the classical propositional logic **Cl** and axiomatized by a formula with a single proposition letter is a sublogic of **KC** [Nishimura, 1960], our theorem also implies the following:

Corollary

Let $L = \mathbf{Int} + \varphi$, where φ is a formula with a single proposition letter, and let $L \subset \mathbf{Cl}$. Then, $\mathbf{Q}L_{wfin}$ and $\mathbf{Q}L.\mathbf{cd}_{wfin}$ are Π^0_1-hard in languages with three individual variables and a single monadic predicate letter.

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Thank you!