TILING PROBLEMS AND COMPLEXITY OF LOGICS

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1. Introduction

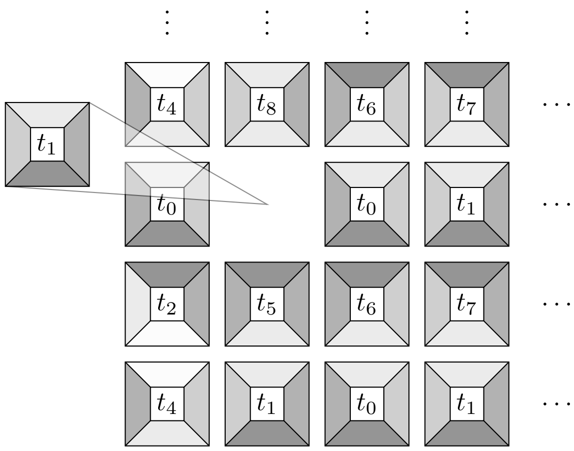
Domino, or tiling, problems [1, 9] provide us with a rich tool allowing to estimate bounds for computational complexity of problems arising in different fields of mathematics, in particular, in algebra [4, 10] and mathematical logic [2, 8, 14, 11, 6]. Sometimes, properties of tilings of some kind can be quite easily expressed in a formal language, and their description can be more elegant than, say, of Turing machines (or other computational models). Indeed, to describe a tiling, we only have to say that, for every tile, there are appropriate tiles on the top and on the right, and that moving right-top or top-right we see the same tile, while for a Turing machine, to describe just a configuration on some step of computation, we have to describe a head position, a state, and symbols stored in tape cells.

Here, we consider two tiling problems, known to be, respectively, Π^0_1-complete and Σ^1_1-complete, and show examples of their simulation in first-order theories and logics whose langages are enriched with some extra expressive means [8] but restricted in the number of individual variables, the number of predicate letters, and their arity.

2. Tiling problems we consider

We may think of a tile as a colored 1×1 square, with a fixed orientation. Each edge is colored. A tile type t consists of a specification of a color for each edge; we write ◧t, ◨t, ⬒t, and ⬓t for the colors of, respectively, the left, the right, the top, and the bottom edges of the tiles of type t.

Let $T = \{t_0, \ldots, t_n\}$ be a set of tile types. Informally, a T-tiling is an arrangement of tiles, whose types are in T, on a grid so that the edge colors of the adjacent tiles match, both horizontally and vertically; see the picture below (tile-holders are in the left-bottom corners of tiles).



The fist tiling problem we consider is the following: given a set $T = \{t_0, \ldots, t_n\}$ of tile types, we are to determine whether there exists a T-tiling $f\colon \mathbb{N} \times \mathbb{N} \to T$ such that, for every $i, j \in \mathbb{N}$,

(1) ◨$f(i,j) =$ ◧$f(i+1,j)$;
(2) ⬒$f(i,j) =$ ⬓$f(i,j+1)$.

This problem is Π^0_1-complete [1]. The second tiling problem we consider can be obtained from the first one by adding an extra requirement

(3) the set $\{j \in \mathbb{N} : f(0,j) = t_0\}$ is infinite,

i.e., claiming that there are infinitely many tiles of type t_0 in the leftmost column. This problem is Σ^1_1-complete [9].

3. Classical theories

Assume, for simplicity, a classical first-order language with an infinite supply of monadic predicate letters $P_0, P_1, P_2, \ldots$ and two binary predicate letters H and V. The intending meaning of $P_k(x)$ is "x is placed with a tile of type t_k"; also, $H(x,y)$ means "y is to the right of x", and $V(x,y)$ means "y is above x". To describe an $\mathbb{N}\times\mathbb{N}$ grid, it is sufficient to say

$$\forall x \exists y\, H(x,y), \quad \forall x \exists y\, V(x,y), \quad \forall x \forall y\, (\exists z\, (H(x,z) \wedge V(z,y)) \leftrightarrow \exists z\, (V(x,z) \wedge H(z,y))).$$

Then, we can say that we are given a T-tiling:

- Each tile-holder holds a unique tile: $\forall x \bigvee\limits_{i=0}^{n} (P_i(x) \wedge \bigwedge\limits_{j\neq i} \neg P_j(x)).$
- The condition (1) for a T-tiling is satisfied: $\forall x \bigwedge\limits_{i=0}^{n} (P_i(x) \to \forall y\, (H(x,y) \to \bigvee\limits_{\text{◨} t_i = \text{◧} t_j} P_j(y))).$
- The condition (2) for a T-tiling is satisfied: $\forall x \bigwedge\limits_{i=0}^{n} (P_i(x) \to \forall y\, (V(x,y) \to \bigvee\limits_{\text{⬒} t_i = \text{⬓} t_j} P_j(y))).$

It is not hard to see that the conjunction of the above formulas is satisfiable if, and only if, there exists a T-tiling $f\colon \mathbb{N}\times\mathbb{N} \to T$ satisfying conditions (1) and (2). As a result, the Church's theorem [3] for the classical first-order logic follows. Since we can simulate all the predicate letters with a single binary one without adding extra individual variables [15, 16], this gives us a short proof of the known refinement [23] of the Church's theorem: the satisfiability problem is undecidable for languages with a single binary predicate letter and three individual variables. Moreover, we readily obtain undecidability (Σ^0_1-hardness) for infinite classes of theories of a binary predicate, again, with three individual variables [15, 16].

Observe that, with the use of Compactness theorem, the existence of a T-tiling satisfying (1) and (2) is equivalent to the existance, for every $n \in \mathbb{N}$, of an $n \times n$ tiling with T-tiles satisfying (1) and (2) for all appropriate i and j. Therefore, we can use only finitely many tile-holders (but their number must be unbounded). This observation allows us to simulate T-tilings on finite models and, thus, to obtain the Trakhtenbrot's theorem [24, 25] for satisfiability over finite models. Again, modulo some linguistic machinations, we obtain undecidability (Π^0_1-harness) for large classes of theories of a binary predicate defined by infinite classes of finite models [15, 16].

Notice that undecidability of some the theories — both Σ^0_1-hardness and Π^0_1-hardness — follow also from proofs like in [5, 13] by means of a general technique described in [22].

4. Classical theories with extra non-elementary expressive means

Having enriched the language with equality and the operator of transitive closure, we can use the transitive closure V^+ of V allowing us to express (3):

$$\exists x \forall y\, (V^+(x,y) \to \exists z\, (z \neq y \wedge V^+(y,z) \wedge P_0(z))).$$

Notice that equality can be eliminated if we add the condition of irreflexivity, i.e., $\forall x\, \neg V(x,x)$; also, variable z can be replaced with x. Then, adding the operator of composition $\circ$ of binary relations, we are able to express that moving right-top and top-right, we see the same tile, using the formula $\forall x \forall y\, ([V \circ H](x,y) \leftrightarrow [H \circ V](x,y))$, which contains only two individual variables. Again, using additional techniques, we can prove that the satisfiability for languages with a single binary relation, equality, the operators of transitive closure and composition is Σ^1_1-hard even for formulas with two variables [15]. Sometimes, the operator of transitive closure can be replaced with the operator asserting the transitivity of a binary relation [16].

5. Some remarks and further results

Examples of the use of tiling problems for obtaining results on the algorithmic complexity of various logics, both propositional and predicate, can be found in [2, 14, 6, 11, 19, 20, 21, 17, 18]. In particular, the tiling problems considered here can be used to obtain complexity results for theories of trees [18] and to prove that modal predicate logics whose Kripke frames are Noetherian orders are Π^1_1-hard in rather poor languages [17]; the latter result gives us an alternate argument for Kripke incompleteness of the predicate counterpart of the Gödel–Löb logic **GL** [12].

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