ON THE QUANTIFIED VERSION OF THE BELNAP–DUNN MODAL LOGIC AND SOME EXTENSIONS OF IT

ALEXANDER V. GREFENSHTEIN

The propositional Belnap–Dunn modal logic BK was introduced in [1]; the lattice of its extensions has been studied in [2, 3, 4]. It can be viewed as conservatively expanding the least normal modal logic K by adding 'strong negation' ($\sim$), which allows us to deal with 'gaps' (incomplete information) and 'gluts' (inconsistent information). On the other hand, BK can be obtained by adding material implication ($\to$) and the absurdity constant ($\bot$) to the logic $\mathsf{K}_{\mathsf{FDE}}$ (see [5]). The latter is the least modal expansion of the Belnap–Dunn four-valued logic, also known as *first-degree entailment* and denoted by FDE (see [6, 7]). Intuitively, the semantic values used in FDE are:

(1) $\mathtt{T}$, which intuitively stands for 'true';
(2) $\mathtt{F}$, which intuitively stands for 'false';
(3) $\mathtt{N}$, which intuitively stands for 'neither true nor false';
(4) $\mathtt{B}$, which intuitively stands for 'both true and 'false'.

Following [1], we present BK in the language

$$\mathcal{L} := \{\wedge, \vee, \to, \Box, \Diamond, \sim, \bot\}\,.$$

Below we use the following abbreviations: $\neg\varphi$ stands for $\varphi \to \bot$ and $\varphi \Leftrightarrow \psi$ ('strong equivalence') stands for $(\varphi \leftrightarrow \psi) \wedge (\sim \varphi \leftrightarrow \sim \psi)$. The corresponding deductive system includes the following axiom schemata:

CL. all the schemata of propositional classical logic in the language $\{\wedge, \vee, \to, \bot\}$;
K1. $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$;
K2. $\Box(\varphi \to \varphi)$;
M1. $\neg\Box\varphi \leftrightarrow \Diamond\neg\varphi$;
M2. $\neg\Diamond\varphi \leftrightarrow \Box\neg\varphi$;
M3. $\Box\varphi \Leftrightarrow \sim \Diamond \sim \varphi$;
M4. $\Diamond\varphi \Leftrightarrow \sim \Box \sim \varphi$;
SN1. $\sim\sim \varphi \leftrightarrow \varphi$;
SN2. $\sim (\varphi \to \psi) \leftrightarrow (\varphi \wedge \sim \psi)$;
SN3. $\sim (\varphi \vee \psi) \leftrightarrow (\sim \varphi \wedge \sim \psi)$;
SN4. $\sim (\varphi \wedge \psi) \leftrightarrow (\sim \varphi \vee \sim \psi)$;
SN5. $\sim \bot$.

As for the rules, we have *modus ponens* and the monotonicity rules for $\Box$ and $\Diamond$:

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ (\mathtt{MP})\,; \qquad \frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}\ (\mathtt{MB})\,; \qquad \frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}\ (\mathtt{MD})\,.$$

By BK we mean the least set of $\mathcal{L}$-formulas containing the axioms above and closed under the rules MP, MB, MD.

Using the canonical models method, it was proved in [1] that BK and some natural extensions of it are strongly complete w.r.t. appropriate Kripke-style semantics. For instance, a BK-*model* is a triple $\langle \mathcal{W}, v^+, v^- \rangle$ that consists of a standard Kripke frame $\mathcal{W}$ with two independent propositional valuations v^+, v^-. Intuitively, v^+ and v^- correspond to verification and falsification respectively. So we have two relations $\Vdash^+$ and $\Vdash^-$ such that for every propositional variable p:

$$\mathcal{M}, w \Vdash^+ p \ :\Longleftrightarrow\ w \in v^+(p)\,;$$
$$\mathcal{M}, w \Vdash^- p \ :\Longleftrightarrow\ w \in v^-(p)\,.$$

The logical connective $\sim$ is treated as a switch between the processes of verifiability and falsifiability:

$$\mathcal{M}, w \Vdash^+ \sim \varphi \ :\Longleftrightarrow\ \mathcal{M}, w \Vdash^- \varphi;$$
$$\mathcal{M}, w \Vdash^- \sim \varphi \ :\Longleftrightarrow\ \mathcal{M}, w \Vdash^+ \varphi.$$

Since in general no restrictions are imposed on the relationship between verifiability and falsifiability, BK turns out to be paraconsistent. Locally, the truth in BK is four-valued. As well as in FDE, these four values are T, F, N, and B.

It is important that some natural BK-extensions are closely related to Nelson's constructive logics N3 (see [8, 9]) and $N4^{\perp}$. The latter is a modification proposed by S. P. Odintsov of the logic N4 originally considered in [10]. It was proved in [1] that N3 and $N4^{\perp}$ can be faithfully embedded into the Belnapian version BS4 of the modal logic S4 and its three-valued extension B3S4 respectively by providing an analogue of the Gödel–McKinsey–Tarski translation of intuitionistic logic into S4.

Although the propositional version of BK is quite well studied, nothing has been previously known about its quantified version. Initially, strong negation appeared in [8] in the first-order setting. It was an approach to an alternative interpretation of negation in intuitionistic arithmetic. This approach was supposed to make intuitionistic negation more constructive, similar to the procedure of falsification instead of the reduction to absurdity. Such arithmetic is now known as Nelson's arithmetic, NA, defined no longer on the basis of intuitionistic first-order logic, but on the basis of Nelson's first-order logic QN3 (see [11]). Since many non-modal logics can be translated into suitable modal logics, the natural question arises: can we do the same for Nelson's logics? Such faithful embeddings can help us in the study of some metamathematical properties of Nelson's logics through the transfer of such properties from the logics in which they are embedded. As we know from the previous paragraph, it can be done in the propositional case. But since QN3 and $QN4^{\perp}$ are first-order (see [12] for $QN4^{\perp}$), we would like to find suitable first-order modal logics in which QN3 and $QN4^{\perp}$ can be embedded.

It would be very natural to expect that the connection between BK-extensions and Nelson's logics should be appropriately inherited to the first-order case. But since the argument of S. P. Odintsov and H. Wansing is based on the completeness of BK-extensions, our first goal is to prove the completeness theorem for the quantified version of BK.

Let σ be a first-order signature. From now on by *σ-formulas* we mean expressions built up from the atomic σ-formulas using the connective symbols of $\mathcal{L}$ and the quantifier symbols $\forall, \exists$ in the usual way. Below we use capital letters for σ-formulas in order to emphasize the difference between the propositional and quantified settings. The deductive system for QBK extends the deductive system for BK by adding the following four axiom schemata:

Q1. $\forall x\, \Phi \to \Phi\,(x/t)$, where t is free for x in Φ;
Q2. $\Phi\,(x/t) \to \exists x\, \Phi$, where t is free for x in Φ;
Q3. $\sim \forall x\, \Phi \leftrightarrow \exists x \sim \Phi$;
Q4. $\sim \exists x\, \Phi \leftrightarrow \forall x \sim \Phi$,

and the Bernays rules:

$$\frac{\Phi \to \Psi}{\Phi \to \forall x\, \Psi}\ (\mathtt{BR1})\,; \qquad \frac{\Phi \to \Psi}{\exists x\, \Phi \to \Psi}\ (\mathtt{BR2})\,.$$

The axioms Q3 and Q4 were originally considered in Nelson's work on constructive arithmetic [8] and were further used in works on Nelson's first-order logics $QN4^{\perp}$ and QN3. Now denote by Form_σ the set of all σ-formulas and by Sent_σ the set of all σ-sentences (which are formulas with no free variable occurrences, as usual). Given $\Gamma \subseteq \mathrm{Sent}_\sigma$ and $\Delta \subseteq \mathrm{Form}_\sigma$, we write $\Gamma \vdash_{\mathsf{QBK}} \Delta$ if there is $\Delta' \subseteq \Delta$ such that the disjunction of Δ' can be obtained from the elements of $\Gamma \cup \mathsf{QBK}$ by means of MP, BR1, and BR2.

One of the specific features of both Nelson's logics and QBK is the lack of closure under the usual replacement rule. However, QBK, as well as BK, is closed under the so-called *weak replacement rule*, which is rendered as

$$\frac{\Psi \Leftrightarrow \Phi}{\Theta\,(\Phi/\Psi) \Leftrightarrow \Theta}\ (\mathtt{WR})\,.$$

It is also notable that the negative normal form theorem remains true for QBK, i.e. for any σ-formula Φ there is a σ-formula $\overline{\Phi}$, in which $\sim$ stands only before atomic subformulas, such that $\Phi \Leftrightarrow \overline{\Phi} \in \mathsf{QBK}$ (cf. [13]).

QBK-*models* are first-order Kripke models with expanding domains in which there are two structures with the same domain at each world w: $\mathfrak{A}_w^+$ for verifiability, and $\mathfrak{A}_w^-$ for falsifiability. For any QBK-model $\mathcal{M}$, any world w of this model, and any σ-sentence Φ, the relations $\mathcal{M}, w \Vdash^+ \Phi$ and $\mathcal{M}, w \Vdash^- \Phi$ are defined by induction on the complexity of Φ. If Φ is atomic, we checking its verifiability and falsifiability at the world w by looking at $\mathfrak{A}_w^+$ and $\mathfrak{A}_w^-$ respectively. The case of strong negation is considered as above. Below are the remaining cases:

$$
\begin{aligned}
\mathcal{M}, w \Vdash^+ \Phi \wedge \Psi \quad &:\Longleftrightarrow \quad \mathcal{M}, w \Vdash^+ \Phi \text{ and } \mathcal{M}, w \Vdash^+ \Psi;\\
\mathcal{M}, w \Vdash^- \Phi \wedge \Psi \quad &:\Longleftrightarrow \quad \mathcal{M}, w \Vdash^- \Phi \text{ or } \mathcal{M}, w \Vdash^- \Psi;\\
\mathcal{M}, w \Vdash^+ \Phi \vee \Psi \quad &:\Longleftrightarrow \quad \mathcal{M}, w \Vdash^+ \Phi \text{ or } \mathcal{M}, w \Vdash^+ \Psi;\\
\mathcal{M}, w \Vdash^- \Phi \vee \Psi \quad &:\Longleftrightarrow \quad \mathcal{M}, w \Vdash^- \Phi \text{ and } \mathcal{M}, w \Vdash^- \Psi;\\
\mathcal{M}, w \Vdash^+ \Phi \to \Psi \quad &:\Longleftrightarrow \quad \mathcal{M}, w \nVdash^+ \Phi \text{ or } \mathcal{M}, w \Vdash^+ \Psi;\\
\mathcal{M}, w \Vdash^- \Phi \to \Psi \quad &:\Longleftrightarrow \quad \mathcal{M}, w \Vdash^+ \Phi \text{ and } \mathcal{M}, w \Vdash^- \Psi;\\
\mathcal{M}, w \Vdash^+ \Box\Phi \quad &:\Longleftrightarrow \quad \text{for each } u \in W, \text{ if } wRu, \text{ then } \mathcal{M}, w \Vdash^+ \Phi;\\
\mathcal{M}, w \Vdash^- \Box\Phi \quad &:\Longleftrightarrow \quad \text{there exists } u \in W \text{ such that } wRu \text{ and } \mathcal{M}, w \Vdash^- \Phi;\\
\mathcal{M}, w \Vdash^+ \Diamond\Phi \quad &:\Longleftrightarrow \quad \text{there exists } u \in W \text{ such that } wRu \text{ and } \mathcal{M}, w \Vdash^+ \Phi;\\
\mathcal{M}, w \Vdash^- \Diamond\Phi \quad &:\Longleftrightarrow \quad \text{for each } u \in W, \text{ if } wRu, \text{ then } \mathcal{M}, w \Vdash^- \Phi;\\
\mathcal{M}, w \Vdash^+ \forall x\, \Phi \quad &:\Longleftrightarrow \quad \text{for each } a \in A_w \text{ it holds that } \mathcal{M}, w \Vdash^+ \Phi\,(x/a)\,;\\
\mathcal{M}, w \Vdash^- \forall x\, \Phi \quad &:\Longleftrightarrow \quad \mathcal{M}, w \Vdash^- \Phi\,(x/a) \text{ for some } a \in A_w;\\
\mathcal{M}, w \Vdash^+ \exists x\, \Phi \quad &:\Longleftrightarrow \quad \mathcal{M}, w \Vdash^+ \Phi\,(x/a) \text{ for some } a \in A_w;\\
\mathcal{M}, w \Vdash^- \exists x\, \Phi \quad &:\Longleftrightarrow \quad \text{for each } a \in A_w \text{ it holds that } \mathcal{M}, w \Vdash^- \Phi\,(x/a)\,.
\end{aligned}
$$

The semantic consequence for QBK is defined in the standard way, using the relation $\Vdash^+$. We prove that QBK is strongly complete w.r.t. this Kripke-style semantics:

Theorem 1. *For any* $\Gamma \subseteq \mathrm{Sent}_\sigma$ *and* $\Delta \subseteq \mathrm{Form}_\sigma$*,*

$$\Gamma \vdash_{\mathsf{QBK}} \Delta \quad \Longleftrightarrow \quad \Gamma \vDash_{\mathsf{QBK}} \Delta.$$

The proof of strong completeness of QBK is carried out by modifying the canonical models method for first-order modal logics (see [14]). In fact, the resulting modification can be successfully applied to prove the strong completeness of some others quantified modal logics with strong negation. Among the interesting BK-extensions are:

(1) those obtained by excluding either gaps, gluts or both;
(2) those obtained by imposing restrictions (expressible by modal formulas) on accessibility relations in Kripke frames.

Syntactically, the following axiom schemata correspond to the exclusion of gaps and gluts respectively:

`ExM`. $\varphi \vee \sim \varphi$;
`Exp`. $\sim \varphi \to (\varphi \to \psi)$.

Intuitively, `ExM` is for excluded middle, and `Exp` is for explosion. As for extensions of the second kind, they are obtained by adding standard axiom schemata expressing suitable properties of accessibility relations. In particular, denote QBK + {`Exp`} and QBK + {`ExM`} by QB3K and QBK° respectively. We prove the following

Theorem 2. QB3K *and* QBK° *are strongly complete w.r.t. appropriate Kripke-style semantics.*

For the quantified analogs of extensions of the second kind we also obtain similar results. In particular, we prove the strong completeness theorems for the quantified analogs of BS4 and B3S4. Let us denote these by QBS4 and QB3S4 respectively.

In conclusion, we generalize the result that Nelson's logics can be faithfully embedded into appropriate BK-extensions to the first-order setting as follows:

Theorem 3. *The logics* QN3 *and* $\mathsf{QN4}^\perp$ *can be faithfully embedded into* QB3S4 *and* QBS4 *respectively.*

Acknowledgements. Taking this opportunity, I would like to thank my advisor, Stanislav O. Speranski, for all his help, guidance and advice.

References

[1] S. P. Odintsov, and H. Wansing. Modal logics with Belnapian truth values. *Journal of Applied Non-Classical Logics*, 20(3), 279–301, 2010.
[2] S. P. Odintsov, and E. I. Latkin. BK-lattices. Algebraic semantics for Belnapian modal logics. *Studia Logica*, 100(1-2), 319–338, 2012.

[3] S. P. Odintsov, and S. O. Speranski. The lattice of Belnapian modal logics: special extensions and counterparts. *Logic and Logical Philosophy*, 25(1), 3–33, 2016.
[4] S. P. Odintsov, and S. O. Speranski. Belnap–Dunn modal logics: truth constants vs. truth values. *Review of Symbolic Logic*, 13(2), 416–435, 2020.
[5] G. Priest. Many-valued modal logics: A simple approach. *Review of Symbolic Logic*, 1(2), 190–203, 2008.
[6] N. Belnap. A useful four-valued logic. In Dunn, J. M., & Epstein, G., editors. *Modern Uses of Multiple-Valued Logic, Volume 2 of Episteme,* Reidel, D., 8–37, 1977.
[7] J. M. Dunn. Intuitive semantics for first-degree entailments and 'coupled trees'. *Philosophical Studies*, 29(3), 149–168, 1976.
[8] D. Nelson. Constructible falsity. *Journal of Symbolic Logic*, 14(1), 16–26, 1949.
[9] N. N. Vorob'ev. A constructive propositional logic with strong negation (in Russian), *Doklady Akademii Nauk SSSR*, 85(3), 465–468, 1952.
[10] A. Almukdad, and D. Nelson. Constructible falsity and inexact predicates. *Journal of symbolic logic*, 49(1), 231–233, 1984.
[11] Y. Gurevich. Intuitionistic logic with strong negation. *Studia Logica* 36(1), 49–59, 1977.
[12] S. P. Odintsov, and H. Wansing. Inconsistency-tolerant description logic: Motivation and basic systems. In Hendricks, V. F., & Malinowski, J., editors. *Trends in Logic: 50 Years of Studia Logica*. Kluwer Academic Publishers, 301–335, 2003.
[13] S. O. Speranski. Modal bilattice logic and its extensions. *Algebra and Logic*, 60(6), 407-424, 2022.
[14] D. M. Gabbay, V. B. Shehtman, and D. P. Skvortsov. *Quantification in Nonclassical Logic, Volume 1*. Elsevier, 2009.

HSE University, Moscow, Russia
Email address: aleksandrgrefenstejn@gmail.com