Topological semantics of the predicate modal calculus QGL extended with non-well-founded proofs *

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The Gödel-Löb provability logic GL is a well-known propositional system whose language, along with Boolean connectives, contains an additional modal connective $\Box$. According to its arithmetical interpretation, the connective $\Box$ corresponds to the standard provability predicate *"... is provable in Peano arithmetic* PA*"*. As shown by Solovay [7], a formula is a theorem of GL if and only if each arithmetical translation of the formula is a theorem of PA. In other words, GL captures the properties of formal provability of PA that are provable in PA itself.

The provability logic GL can also be described in terms of its relational semantics. This system is complete with respect to the class of irreflexive transitive Kripke frames without infinite ascending chains. However, GL is only weakly complete for its relational interpretation.

Strong completeness is achieved if one considers topological (or neighbourhood) semantics of the given logic. The class of topological spaces corresponding to GL consists of all scattered topological spaces (X, τ), where the modal connective $\Box$ is interpreted as the co-derived-set operator $cd_\tau(Y) = \{x \in X \mid \exists U \in \tau\, (x \in U \wedge U \smallsetminus \{x\} \subset Y)\}$.

An interesting feature of GL is that this logic allows cyclic and non-well-founded reasoning. In [3, 1], it was shown that GL can be defined by means of a sequent calculus allowing non–well-founded proofs. In [4, 5], the standard axiomatic calculus for GL was extended with non–well-founded derivations and various topological completeness results for the obtained system were established.

In the present talk, we focus on a first-order predicate version of GL denoted by QGL. We consider this system in a language without function symbols and constants (to simplify technical details) and define it by the following axioms and inference rules.

Axioms:

- tautologies of classical propositional logic,
- $\Box(A \to B) \to (\Box A \to \Box B)$,
- $\Box A \to \Box\Box A$,
- $\Box(\Box A \to A) \to \Box A$,
- $\forall x\, A(x) \to A(y)$,
- $\forall x\, (A \to B) \to (A \to \forall x\, B)$, where $x \notin FV(A)$.

Inference rules:

$$\mathsf{mp}\,\frac{A \qquad A \to B}{B}\,, \qquad \mathsf{nec}\,\frac{A}{\Box A}\,, \qquad \mathsf{gen}\,\frac{A}{\forall x\, A}\,.$$

*The work was partially supported by the HSE Academic Fund Programme (Project: Research and Study Group number 23-00-022).

Thanks to Montagna [2], we know that QGL is not arithmetically complete. He also showed that this system is not complete with respect to its Kripke semantics. However, it is not something out of the ordinary. In many cases, predicate versions of Kripke complete modal propositional systems are incomplete for their relational interpretations. Whether QGL is topologically complete, we do not know, but we conjecture that it is not.

We introduce an extension of QGL obtained by allowing non-well-derivations in the QGL calculus. A *non-well-founded derivation, or ∞-derivation,* is a (possibly infinite) tree whose nodes are marked by predicate modal formulas and that is constructed according to the rules (mp), (gen) and (nec). In addition, any infinite branch in an ∞-derivation must contain infinitely many applications of the rule (nec). Below is an example of an ∞-derivation:

$$\mathsf{mp}\,\dfrac{\mathsf{nec}\,\dfrac{\mathsf{gen}\,\dfrac{\mathsf{mp}\,\dfrac{\begin{matrix}\vdots\\ \Box\forall x_2\, P_2(x_2)\end{matrix} \qquad \Box\forall x_2\, P_2(x_2)\to P_1(x_1)}{P_1(x_1)}}{\forall x_1\, P_1(x_1)}}{\Box\forall x_1\, P_1(x_1)} \qquad \Box\forall x_1\, P_1(x_1)\to P_0(x_0)}{P_0(x_0)}\,.$$

A *non-well-founded proof, or ∞-proof*, is an ∞-derivation, where all leaves are marked by axioms of QGL. We write $\mathsf{QGL}_\infty \vdash A$ if there is an ∞-proof with the root marked by A.

The relationship between QGL and QGL$_\infty$ can be explained as follows: QGL corresponds to the fragment QGL$_\infty$, which is obtained if, instead of arbitrary non-well-founded derivations, we restrict ourselves to the so-called cyclic derivations (more precisely, to their unfoldings).

Our main result is that QGL$_\infty$ is complete with respect to the class of predicate topological frames for QGL$_\infty$ with constant domains. We define a *predicate topological frame for* QGL$_\infty$ as a tuple (X,τ,D), where (X,τ) is a scattered topological space and D is a non-empty domain. Note that, in the case of topological semantics, the constant domain condition does not imply validity of the Barcan formula in contrast to the case of relational frames.

Let us recall some basic notions of semantics of predicate modal systems. A *valuation in* D is a function sending each n-ary predicate letter to an n-ary relation on D, and a *variable assignment* is a function from the set of variables $Var = \{x_0, x_1, x_2, \dots\}$ to the domain D. For QGL$_\infty$, a *predicate topological model* $\mathcal{M} = (X,\tau,D,\xi)$ is a predicate topological frame (X,τ,D) of QGL$_\infty$ together with an indexed family of valuations $\xi = (\xi_w)_{w\in X}$ in D. Elements of the set X are usually called *worlds* of the model.

The *truth of a formula A at a world w of a model $\mathcal{M}$ under a variable assignment h* is defined as

- $\mathcal{M}, w, h \nvDash \bot$,
- $\mathcal{M}, w, h \vDash P(x_1,\dots,x_n) \Longleftrightarrow (h(x_1),\dots,h(x_n)) \in \xi_w(P)$,
- $\mathcal{M}, w, h \vDash A\to B \Longleftrightarrow \mathcal{M}, w, h \nvDash A$ or $\mathcal{M}, w, h \vDash B$,
- $\mathcal{M}, w, h \vDash \Box A \Longleftrightarrow \exists U\in\tau\,(w\in U$ and $\forall w'\in U\smallsetminus\{w\}\ \mathcal{M}, w', h \vDash A)$,
- $\mathcal{M}, w, h \vDash \forall x\, A \Longleftrightarrow \mathcal{M}, w, h' \vDash A$ for any varible assignment h' such that $h' \stackrel{x}{=} h$,

where $h' \stackrel{x}{=} h$ means that $h'(y) = h(y)$ for each $y \in Var\smallsetminus\{x\}$.

A *formula A is true in $\mathcal{M}$* if A is true at all worlds of $\mathcal{M}$ under all variable assignments. In addition, *A is valid in a frame $\mathcal{F}$* if A is true in all models over $\mathcal{F}$.

Theorem 1 (topological completeness). *For any formula A, $\mathsf{QGL}_\infty \vdash A$ if and only if A is valid in every predicate topological frame of QGL_∞.*

In order to obtain this result, we focus on a proof-theoretic presentation of QGL_∞ in a form of a sequent calculus allowing non–well-founded proofs. This calculus is defined by the following initial sequents and inference rules:

$$\Gamma, P(\vec{x}) \Rightarrow P(\vec{x}), \Delta, \qquad \Gamma, \bot \Rightarrow \Delta,$$

$$\to_{\mathsf{L}} \frac{\Gamma, B \Rightarrow \Delta \qquad \Gamma \Rightarrow A, \Delta}{\Gamma, A \to B \Rightarrow \Delta}, \qquad \to_{\mathsf{R}} \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta},$$

$$\forall_{\mathsf{L}} \frac{\Gamma, A(y), \forall x\, A \Rightarrow \Delta}{\Gamma, \forall x\, A \Rightarrow \Delta}, \qquad \forall_{\mathsf{R}} \frac{\Gamma \Rightarrow A(y), \Delta}{\Gamma \Rightarrow \forall x\, A, \Delta}\ (y \notin FV(\Gamma \cup \Delta)),$$

$$\Box \frac{\Gamma, \Box\Gamma \Rightarrow A}{\Pi, \Box\Gamma \Rightarrow \Box A, \Delta}.$$

In addition, every infinite branch in a non-well-founded proof of this calculus must contain infinitely many applications of the rule ($\Box$).

Our proof of Theorem 1 is inspired by two other sequent-based completeness proofs. We follow the proof for classical predicate logic based on reduction trees and the proof for the system GL extended with non-well-founded derivations from [5].

We also establish a strong version of Theorem 1. We write $\Gamma \vDash A$ if for any predicate topological model $\mathcal{M} = (X, \tau, D, \xi)$, any world w of $\mathcal{M}$ and any variable assignment $h\colon Var \to D$

$$\forall B \in \Gamma\ \mathcal{M}, w, h \vDash B \Longrightarrow \mathcal{M}, w, h \vDash A.$$

Combining the standard ultaproduct costruction and Shehtman's ultrabouquet construction from [6], we obtain the following proposition.

Proposition 2 (compactness). *If $\Gamma \vDash A$, then there is a finite subset Γ_0 of Γ such that $\Gamma_0 \vDash A$.*

Applying the previous results, we prove strong completeness for the following syntactic consequence relation. We put $\Gamma \vdash A$ if there is an ∞-derivation δ with the root marked by A such that, for each leaf a of δ that is not marked by an axiom, a is marked by a formula from Γ, and there are no applications of the rules (gen) and (nec) on the path from the root of δ to the leaf a.

Corollary 3 (strong completeness). *For any set of formulas Γ and any formula A,*

$$\Gamma \vdash A \Longleftrightarrow \Gamma \vDash A.$$

In conclusion, we note one interesting question: what is the complexity of QGL_∞? It may well turn out that this system is not computably enumerable. In the area of predicate provability logic, there is an example. According to Vardanian's result, the set of predicate modal formulas provable in PA under any interpretation is Π^0_2-complete [8, 9].

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