LINEAR ORDERS INTERPRETABLE IN PRESBURGER ARITHMETIC

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This talk is based on a joint work with Fedor Pakhomov.

We consider the interpretations of linear orders in Presburger arithmetic. Let a condensation cL of a linear order $(L, <)$ be the result of identifying the points with finite distance between each other, and the VD^*-rank of L the least ordinal α such that α iterations of condensation produce a finite linear order, if such α exists. As we have previously shown in [1],

Theorem 1. *Each linear order interpretable m-dimensionally in Presburger arithmetic has the VD^*-rank m or below.*

In particular, all linear orders interpretable in Presburger arithmetic are scattered, that is, do not contain a suborder isomorphic to $\mathbb{Q}$.

However, there are uncountably many non-isomorphic scattered linear orders of VD^*-rank 2 but only countably many linear orders can be interpretable in Presburger arithmetic. Hence, already for rank 2 the necessary condition of Theorem 1 is not sufficient. A challenge emerges to give a complete classification of linear orders interpretable m-dimensionally in Presburger arithmetic. We have established the following:

Theorem 2 ([2])**.** *A linear ordering $(L, <)$ is m-dimensionally interpretable in Presburger arithmetic for some $m \geq 1$ iff there exists some $k \in \mathbb{N}$ and a Presburger-definable set $D \subseteq \mathbb{Z}^k$ such that L is isomorphic to the restriction of the lexicographic ordering on $\mathbb{Z}^k$ onto D.*

We obtain Theorem 2 by considering a more general case of families of linear orders interpretable in Presburger arithmetic. The result can be compared with the results on the embeddability of linear orders into lexicographic orders in o-minimal structures [3]: in particular, linear orders interpretable in the elementary theory of $(\mathbb{N}; =, <)$ are actually embeddable into the lexicographic ordering on $\mathbb{Z}^k$ for some k. Hence, despite the absence of o-minimality, with regards to definability of linear orders, Presburger arithmetic would have some properties analogous to o-minimal theories.

References

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