ON INTERPRETATIONS IN BÜCHI ARITHMETICS

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Büchi arithmetics $\mathbf{BA}_n, n \geq 2$, are the theories $\mathbf{Th}(\mathbb{N}; =, +, V_n)$ where V_n is an additional unary functional symbol such that $V_n(x)$ is the largest power of n that divides x. By convention, we put $V_n(0) = 0$. They are a series of natural extensions of Presburger arithmetic $\mathbf{PrA} = \mathbf{Th}(\mathbb{N}; =, +)$ [7] that are also complete and decidable. The theories $\mathbf{BA}_n$ were proposed by J. Büchi [2] in order to describe the sets of natural numbers recognizable by finite automata through definability in some arithmetical language.

Let $Digit_n(x, y)$ be the digit corresponding to n^y in the n-are expansion of x. We consider automata over the alphabet $\{0, \dots, n-1\}^m$. Assume that at step k, the automaton receives the input $(Digit_n(x_1, k), \dots, Digit_n(x_m, k))$, containing the digits corresponding to n^k in the n-ary expansion of numbers $(x_1, \dots, x_m)$. We say the automaton accepts the tuple $(x_1, \dots, x_m)$ if it accepts the corresponding sequence of tuples $(Digit_n(x_1, k), \dots, Digit_n(x_m, k))$. Then the following well-known result by V. Bruyère ([3, 4]) holds:

Proposition 1. *Let $\varphi(x_1, \dots, x_m)$ be a $\mathbf{BA}_n$-formula. Then there is an effectively constructed automaton $\mathcal{A}$ such that $(a_1, \dots, a_m)$ is accepted by $\mathcal{A}$ iff $\mathbb{N} \models \varphi(a_1, \dots, a_m)$.*

Contrariwise, let $\mathcal{A}$ be a finite automaton working on m-tuples of n-ary natural numbers. Then there is an effectively constructed $\mathbf{BA}_n$-formula $\varphi(x_1, \dots, x_m)$ such that $\mathbb{N} \models \varphi(a_1, \dots, a_m)$ iff $(a_1, \dots, a_m)$ is accepted by $\mathcal{A}$.

Let $\mathbf{T}$ and $\mathbf{U}$ be two first-order theories. We call an m_1-dimensional interpretation ι_1 and m_2-dimensional interpretation ι_2 from $\mathbf{T}$ to $\mathbf{U}$ *provably isomorphic*, if in the language of $\mathbf{U}$ there is a formula $F(\overline{x}, \overline{y})$ expressing an isomorphism f between the corresponding internal models $\mathfrak{A}_1$ and $\mathfrak{A}_2$, and the statement that f is an isomorphism must be provable in $\mathbf{U}$.

Note that two interpretations into the elementary theory of a model $\mathfrak{B}$ are provably isomorphic iff there is an isomorphism between their internal models in $\mathfrak{B}$ expressible in the language of $\mathfrak{B}$. As $\mathbf{BA}_n$ is the elementary theory of the model $(\mathbb{N}; =, +, V_n)$ respectively, it is sufficient to study the interpretations in this model when studying interpretations up to a provable isomorphism.

A. Visser has asked the following question: *given an weak arithmetical theory $\mathbf{T}$ without ability to encode syntax but with full induction, does it hold that each interpretation (one-dimensional or multi-dimensional) of $\mathbf{T}$ into itself is provably isomorphic to the trivial one?* This question was previously answered positively for Presburger arithmetic $\mathbf{PrA}$ in the author's joint work [6] with F. Pakhomov.

We show that

Lemma 2. *Each $\mathbf{BA}_k$ is interpretable in any of $\mathbf{BA}_l$, $k, l \geq 2$.*

Hence, it is sufficient to consider a particular Büchi arithmetic, such as $\mathbf{BA}_2$.

In the talk, we establish that each interpretation of $\mathbf{BA}_n$ in itself is isomorphic to the trivial one [8]. Furthermore, we show this result holds already for the interpretations of Presburger arithmetic in $\mathbf{BA}_n$:

Theorem 3. *Let ι be a (one- or multi-dimensional) interpretation of $\mathbf{PrA}$ in $(\mathbb{N}; =, +, V_n)$. Then the internal model induced by ι is isomorphic to the standard one.*

From Proposition 1 it follows that an algebraic structure is interpretable in $\mathbf{BA}_n$ iff it has an automatic [5] presentation. Hence, in automatic terms, the statement proved implies there is no automatic non-standard model of any Büchi arithmetic.

The proof is based on the contradiction between the Kemeny-style description of the order types of non-standard models of Büchi arithmetics and the following condition on automatic torsion-free abelian groups established by Braun and Strüngmann [1]:

Proposition 4. *Let $(A, +)$ be an automatic torsion-free abelian group. Then there exists a subgroup B of A isomorphic to $\mathbb{Z}^m$ for some natural m such that the orders of the elements in $C = A/B$ are only divisible by a finite number of different primes $p_1, \dots, p_s$.*

This gives a partial positive answer to the question of Visser.

Whether the isomorphism of Theorem 3 is always definable by a $\mathbf{BA}_n$-formula remains a problem for future research.

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