On the system of positive slices in the structure of superintuitionistic predicate logics

Mikhail Rybakov

Higher School of Modern Mathematics MIPT and HSE University, Moscow, Russia

Dmitry Shkatov

School of Computer Science and Applied Mathematics, University of the Witwatersrand, Johannesburg, South Africa

Dmitrij Skvortsov

Federal Research Center "Computer Science and Control", Russian Academy of Sciences, Moscow, Russia

Abstract

We study the system of classes of superintuitionistic predicate logics induced by the equivalence relation identifying logics with the same positive fragment. We call such classes positive slices. We state a condition guaranteeing that logics determined by classes of Kripke frames or Kripke sheaves share a positive fragment, and so belong to the same positive slice. We then use this condition to prove that some well-known superintuitionistic predicate logics have the same positive fragment. We also present an example of a continuum of logics whose positive slices are singletons.

Keywords: superintuitionistic predicate logic, system of slices, positive fragment.

1 Introduction

Studying classes of logics, defined either syntactically or semantically, rather than particular logical systems, has proved a fruitful approach to the study of superintuitionistic and modal *propositional* logics; see, e.g., [1,24] for systematic surveys. A similar approach has also been fruitful in the study of superintuitionistic and modal *predicate* logics; see, e.g., [5,15,4,11]. Often, in particular in [5,15,4,11], the unit of study is then a class of logics lying between two particular system; such classes are often called *intervals* or, as in this paper, *segments*. For example, [11] proves that all superintuitionistic predicate logics between the intuitionistic predicate logic **QH** and the predicate counterpart **QKC** of the propositional logic **KC** of the weak law of the excluded middle are undecidable in languages with two individual variables and a single monadic predicate letter.

Recently, Skvortsov [20], developing ideas of Hosoi [6,7] for propositional logics, proposed to classify superintuitionistic predicate logics into systems of *slices*, which are convex classes induced by an equivalence relations on the class of all logics (we note that segments are convex, and so are subsumed by this approach). This slice-based approach to the study of predicate logics appears to be more powerful than the segment-based one: segments are usually chosen so that all their logics share a common property; thus, by considering slices of all logics sharing a property, rather than particular segments, we might obtain more general, in the sense of the set of the logics covered, results. Obviously, different equivalence relations, and therefore different systems of slices, turn out to be useful for different purposes.

The present paper is meant to be a first sketch of a study of the system of slices induced by the equivalence relation identifying logics with the same positive fragment. This equivalence relation proves useful in, among others, the study of the computational properties of superintuitionistic predicate logics [11,12]. For a study of another system of slices, see [18,20].

In this paper, we obtain (Proposition 5.8) a continuum of singleton positive slices, thus proving the existence of a continuum of logics with a unique positive fragment. On the other hand, we identify, in Theorem 5.12 (Main Theorem), conditions ensuring that logics determined by classes of Kripke frames or Kripke sheaves have the same positive fragment; this allows us to obtain examples of non-trivial positive slices of superintuitionistic predicate logics. Theorem 5.12 generalizes an observation by Yankov [25, Theorem] that the intuitionistic propositional logic $\mathbf{H}$ has the same positive fragment as the propositional logic $\mathbf{KC} = \mathbf{H} + \neg p \vee \neg\neg p$, as well as a similar observation about corresponding predicate logics [13, Proposition 10.2]. Using Theorem 5.12, we, in particular, prove (Propositions 5.14, 5.16, and 5.17) that some well-known superintuitionistic predicate logics have identical positive fragments. We note that, in this context, the use of Kripke sheaf semantics leads to stronger results than the use of the more familiar Kripke frame semantics (see Remark 5.21). We mention important classes of posets to which Theorem 5.12 does not apply; the study of positive fragments of the logics of those posets requires techniques other than those used in this paper.

The paper is structured as follows. Section 2 contains preliminaries on superintuitionistic predicate logics and formulas. In Section 3, we define the system of positive slices in the lattice of superintuitionistic predicate logics and show that the lattice of slices is isomorphic to the lattice of positively axiomatizable logics. In Section 4, we recall Kripke sheaf and Kripke frame semantics for superintuitionistic predicate logics. In Section 5, we present our main results on positive slices. Subsection 5.1 contains an example of a continuum of degenerate (singleton) positive slices. Subsection 5.2 contains our Main Theorem stating conditions guaranteeing that logics determined by classes of Kripke frames or Kripke sheaves have the same positive fragment; it also contains consequences of Main Theorem concerning well-known superintuitionistic predicate logics.

Lastly, in Subsection 5.3, we mention important logics to which Main Theorem does not apply. Section 6 outlines directions for future work.

2 Preliminaries on logics and formulas

We consider logics in a pure (i.e., without individual constants, function symbols, or equality) predicate language $\mathcal{L}$ containing the following symbols: countably many individual variables; for every $n \geqslant 0$, countably many n-ary predicate letters (nullary letters are identified with proposition letters); the propositional constant $\bot$; the binary connectives $\wedge$, $\vee$, and $\rightarrow$; the quantifier symbols $\exists$ and $\forall$. The definition of $\mathcal{L}$-formulas (or, simply, formulas) is standard. We use the standard abbreviations $\neg A = A \rightarrow \bot$ and $A \leftrightarrow B = (A \rightarrow B) \wedge (B \rightarrow A)$, and adopt the usual conventions about omitting parentheses. In what follows, the language $\mathcal{L}$ is identified with the set of its formulas. A formula is *propositional* if it contains no individual variables, and hence no non-nullary predicate letters and no quantifier symbols. The free variables of a formula are its *parameters*. The universal closure of a formula A, which may be assumed to be unique up to the enumeration of A's parameters, is denoted by $\bar{\forall} A$.

2.1 The lattice of superintuitionistic predicate logics

A *superintuitionistic predicate logic*, or simply *logic*, is a set of formulas including the intuitionistic predicate logic $\mathbf{QH}$ and closed under Predicate Substitution, Modus Ponens, and Generalisation; thus, $\mathbf{QH}$ is the smallest superintuitionistic predicate logic. If $\mathbf{L}$ is a logic and A a formula, then $\mathbf{L} \vdash A$ means the same as $A \in \mathbf{L}$. The smallest logic including a logic $\mathbf{L}$ and a set Γ of formulas is denoted by $\mathbf{L} + \Gamma$; if A is a formula, we write $\mathbf{L} + A$ instead of $\mathbf{L} + \{A\}$. The *logical sum* of a family $\{\mathbf{L}_\theta : \theta \in \Theta\}$, where Θ is an index set, of logics is the smallest logic including $\bigcup\{\mathbf{L}_\theta : \theta \in \Theta\}$; notice that the logical sum of the empty family of logics is $\mathbf{QH}$ and that the logical sum of L_1 and L_2 is the logic $L_1 + L_2$. If A and B are formulas and $\mathbf{L}$ a logic, then we say that

- A *implies* B *in* $\mathbf{L}$, and write $A \Rightarrow_{\mathbf{L}} B$, if $\mathbf{L} \vdash A \rightarrow B$;
- B is *derivable* from A, and write $A \vdash B$, if $\mathbf{QH} + A \vdash B$;
- A and B are *equivalent in* $\mathbf{L}$ if $\mathbf{L} \vdash A \leftrightarrow B$;
- A and B are *deductively equivalent* if they are mutually derivable, i.e., if $A \vdash B$ and $B \vdash A$, or, equivalently, if $\mathbf{QH} + A = \mathbf{QH} + B$.

It should be clear that, if A and B are deductively equivalent, then $A \in \mathbf{L}$ if, and only if, $B \in \mathbf{L}$, for every logic $\mathbf{L}$.

It is well known that the set of all logics forms a lattice with respect to the set-theoretic inclusion; we denote this lattice by $\mathfrak{L}$; algebraically, the meet of $\mathbf{L}_1, \mathbf{L}_2 \in \mathfrak{L}$ is the logic $\mathbf{L}_1 \cap \mathbf{L}_2$, and the join of $\mathbf{L}_1, \mathbf{L}_2 \in \mathfrak{L}$ is the logic $\mathbf{L}_1 + \mathbf{L}_2$. The least element of $\mathfrak{L}$ is the logic $\mathbf{QH}$; the greatest, the (absolutely) inconsistent logic $\mathcal{L}$ (the set of all formulas), which, obviously, coincides with both $\mathbf{QH} + p$ and $\mathbf{QH} + \bot$. The lattice $\mathfrak{L}$ is complete since intersections and logical sums of arbitrary (finite or infinite) families of logics are themselves logics.

2.2 Delta-operation on formulas

In Section 5.3, we use the δ-operation on formulas introduced by Hosoi [7]. Here, we recall its definition and the syntactic properties used later on; for more background, consult [4, Section 1.16].

If A is a formula and p a proposition letter not occurring in A, then $\delta A = q \vee (q \to A)$. The following is well known (see, e.g., [4, Lemma 1.16.6]) and easy to check:

Fact 2.1 *For every formulas A and B,*

(1) $\mathbf{QH} \vdash A \to \delta A$.

(2) $\mathbf{QH} \vdash \delta(A \to B) \to (\delta A \to \delta B)$.

(3) $\mathbf{QH} \vdash \delta(A \wedge B) \leftrightarrow (\delta A \wedge \delta B)$.

2.3 Some important formulas

We shall consider the following standard formulas (here, $h < \omega$ and $n < \omega$):

$$
\begin{array}{lll}
J & = & \neg p \vee \neg\neg p; \\
Z & = & (p \to q) \vee (q \to p); \\
CD & = & \forall x\,(P(x) \vee q) \to \forall x\, P(x) \vee q; \\
K & = & \forall x\,\neg\neg P(x) \to \neg\neg\forall x\, P(x); \\
E & = & \neg\neg\exists x\, P(x) \to \exists x\,\neg\neg P(x); \\
JE & = & \neg\exists x\, P(x) \vee \exists x\,\neg\neg P(x); \\
U & = & \forall x \forall y\,(P(x) \to P(y)); \\
U' & = & \exists x\, P(x) \to \forall x\, P(x); \\
P_0 & = & \bot; \\
P_{h+1} & = & \delta P_h \;=\; q_h \vee (q_h \to P_h); \\
P_0^+ & = & \bot; \\
P_{h+1}^+ & = & \forall x\,(Q_h(x) \vee (Q_h(x) \to P_h^+)); \\
Wid_n & = & \bigvee_{i=0}^{n}(p_i \to \bigvee_{j \neq i} p_j).
\end{array}
$$

Thus, P_h is a propositional formula with proposition letters $q_0, \ldots, q_{h-1}$, and P_h^+ is a formula with monadic predicate letters $Q_0, \ldots, Q_{h-1}$. We note that both P_h and P_h^+ are deductively equivalent to positive formulas obtained by replacing occurrences of $\bot$ in them with fresh proposition letters. The following is well known and easy to check:

Fact 2.2

(1) $\mathbf{QH} \vdash U \leftrightarrow U'$*;*

(2) $\mathbf{QH} + JE = \mathbf{QH} + J \wedge E$*;*

(3) $\mathbf{QH} + Z \vdash J$.

Lemma 2.3 $\mathbf{QH} + CD + J \vdash E$.

Proof It is not hard to see that

$$\mathbf{QH} + J \vdash \forall x\,(\neg P(x) \vee \exists x\,\neg\neg P(x)).$$

Hence,

$$\mathbf{QH} + CD + J \vdash \forall x\, \neg P(x) \vee \exists x\, \neg\neg P(x).$$

Since $\mathbf{QH} \vdash \forall x\, \neg P(x) \leftrightarrow \neg\exists x\, P(x)$, it follows that

$$\mathbf{QH} + CD + J \vdash JE.$$

Thus, by Fact 2.2 (2), $\mathbf{QH} + CD + J \vdash E$. □

Corollary 2.4 $\mathbf{QH} + CD + Z \vdash E$.

Proof Immediate from Lemma 2.3 and Fact 2.2 (3). □

Remark 2.5 Without CD, the formula E is not derivable from either J or Z, i.e., neither $\mathbf{QH} + J \vdash E$ nor $\mathbf{QH} + Z \vdash E$; see Lemma 5.20.

Lemma 2.6 $\mathbf{QH} + J + P_2 \vdash Z$*; moreover,* $\mathbf{QH} \vdash J \wedge P_2 \to Z$.

Proof Observe that

$$\begin{aligned}(\neg q \vee \neg\neg q) \wedge (p \vee (p \to q \vee \neg q)) &\Rightarrow_{\mathbf{QH}} \neg q \vee p \vee (p \to q) \\ &\Rightarrow_{\mathbf{QH}} (q \to p) \vee (p \to q).\end{aligned}$$

□

2.4 Positive formulas and positively axiomatizable logics

An $\mathcal{L}$-formula is *positive* if it does not contain occurrences of $\bot$. The set of positive formulas is denoted by $\mathcal{L}^+$. A logic $\mathbf{L}$ is *positively axiomatizable* if it is axiomatizable over $\mathbf{QH}$ only by positive formulas; thus, positively axiomatizable logics are those representable as $\mathbf{QH} + \Gamma$, with $\Gamma \subseteq \mathcal{L}^+$. The set of all positively axiomatizable logics shall be denoted by $\mathfrak{L}_{pos}$.

Proposition 2.7 $\mathfrak{L}_{pos}$ *is a sublattice of* $\mathfrak{L}$.

Proof Obviously, $\mathfrak{L}_{pos}$ is closed under arbitrary (finite and infinite) logical sums. We next show that it is closed under intersections. Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be positively axiomatizable. Then, there exist $\Gamma_1, \Gamma_2 \subseteq \mathcal{L}^+$ such that $\mathbf{L}_i = \mathbf{QH} + \Gamma_i$, for $i \in \{1, 2\}$. By [4, Proposition 2.10.1 (1)],

$$\mathbf{L}_1 \cap \mathbf{L}_2 \;=\; \mathbf{QH} + \{\bar\forall A_1^m \vee \bar\forall A_2^m : A_1 \in \Gamma_1,\; A_2 \in \Gamma_2,\; m \geqslant 0\},$$

where $\bar\forall A^m$ denotes the universal closure of an m-shift [1] of the formula A. Since m-shifts of positive formulas are positive, it follows that $\mathbf{L}_1 \cap \mathbf{L}_2$ is positively axiomatizable. □

Proposition 2.8 *The lattice* $\mathfrak{L}_{pos}$ *is complete.*

[1] An m-shift of a formula A is obtained from A by a substitution of a special form intended to increase arities of predicate letters of A by m, using a fixed list of m fresh variables; for details, see [4, Section 2.5].

Proof As we have seen, $\mathfrak{L}_{pos}$ is a sublattice of $\mathfrak{L}$ with respect to arbitrary logical sums; hence, each family $\{\mathbf{L}_\theta : \theta \in \Theta\}$ of elements of $\mathfrak{L}_{pos}$ has a supremum, its logical sum; therefore, it also has an infimum, which is the logical sum of $\{\mathbf{L} : \mathbf{L} \subseteq \mathbf{L}_\theta, \text{for all } \theta \in \Theta\}$. □

We do not know if the infimum of an infinite family of elements of $\mathfrak{L}_{pos}$ coincides with its intersection:

Problem 2.9 *Is the lattice $\mathfrak{L}_{pos}$ closed under arbitrary intersections?*

We give some examples of logics that are not positively axiomatizable in Subsection 5.4.

3 Positive slices

3.1 Convex sets in the lattice of logics

We say that a set S of the elements of the lattice $\mathfrak{L}$ of logics is *convex* if

$$\forall \mathbf{L}_1, \mathbf{L}_2 \in S \; \forall \mathbf{L}_0 \in \mathfrak{L} \; (\mathbf{L}_1 \subseteq \mathbf{L}_0 \subseteq \mathbf{L}_2 \Rightarrow \mathbf{L}_0 \in S).$$

We shall only be interested in non-empty convex subsets of $\mathfrak{L}$ (the empty set is trivially convex). Special types of convex subsets of $\mathfrak{L}$ are *segments* and *intervals*: if $\mathbf{L}_1, \mathbf{L}_2 \in \mathfrak{L}$ and $\mathbf{L}_1 \subseteq \mathbf{L}_2$, then a segment in $\mathfrak{L}$ is a set

$$[\mathbf{L}_1, \mathbf{L}_2] = \{\mathbf{L} : \mathbf{L}_1 \subseteq \mathbf{L} \subseteq \mathbf{L}_2\},$$

and an interval in $\mathfrak{L}$ is a (possibly empty) set

$$(\mathbf{L}_1, \mathbf{L}_2) = \{\mathbf{L} : \mathbf{L}_1 \subset \mathbf{L} \subset \mathbf{L}_2\}.$$

Notice that for the interval $(\mathbf{L}_1, \mathbf{L}_2)$ to be non-empty it is necessary, but not sufficient, that $\mathbf{L}_1 \subset \mathbf{L}_2$.

It should be clear that a convex set is a segment if, and only if, it contains a least and a greatest element. On the other hand, a set S of the elements of $\mathfrak{L}$ is convex if, and only if, S contains the segment $[\mathbf{L}_1, \mathbf{L}_2]$ whenever $\mathbf{L}_1, \mathbf{L}_2 \in S$.

3.2 Systems of slices in the lattice of logics

A *system* $(S_\theta : \theta \in \Theta)$ *of slices in* $\mathfrak{L}$, where Θ is an index set, is a partition of $\mathfrak{L}$ into convex subsets, called the *slices* of the system. Thus, slices are non-empty, mutually disjoint convex subsets of $\mathfrak{L}$ whose union coincides with $\mathfrak{L}$. It is well known that every such partition is induced by an equivalence on $\mathfrak{L}$.

Of course, not every system of slices is worth studying—either the slices themselves or the corresponding equivalence relation should be meaningful and interesting. In this paper, we consider one such example, a system of positive slices.

3.3 Positive slices and partial ordering on positive slices

3.3.1 Positive fragments and positive slices

The *positive fragment* of a logic $\mathbf{L}$ is the set $\mathbf{L}^+ = \mathbf{L} \cap \mathcal{L}^+$ of its positive formulas.

Lemma 3.1 *If $\mathbf{L}$ is a logic, then $\mathbf{L}^+ = (\mathbf{QH} + \mathbf{L}^+)^+$.*

Proof Clearly, $\mathbf{QH}+\mathbf{L}^+ \subseteq \mathbf{L}$; hence, $(\mathbf{QH}+\mathbf{L}^+)^+ \subseteq \mathbf{L}^+$. On the other hand, since $\mathbf{L}^+ \subseteq \mathbf{QH}+\mathbf{L}^+$, it follows that $\mathbf{L}^+ \subseteq (\mathbf{QH}+\mathbf{L}^+)\cap\mathcal{L}^+ = (\mathbf{QH}+\mathbf{L}^+)^+$. □

We next introduce a preorder on logics with respect to the inclusion of their positive fragments; we also introduce the equivalence induced by this preorder: for every $\mathbf{L}_1, \mathbf{L}_2 \in \mathfrak{L}$, put

$$\begin{array}{lcl}\mathbf{L}_1 \leqslant_{pos} \mathbf{L}_2 & \rightleftharpoons & \mathbf{L}_1^+ \subseteq \mathbf{L}_2^+;\\ \mathbf{L}_1 \equiv_{pos} \mathbf{L}_2 & \rightleftharpoons & \mathbf{L}_1 \leqslant_{pos} \mathbf{L}_2 \text{ and } \mathbf{L}_2 \leqslant_{pos} \mathbf{L}_1.\end{array}$$

Thus, $\mathbf{L}_1 \equiv_{pos} \mathbf{L}_2$ means that $\mathbf{L}_1^+ = \mathbf{L}_2^+$. We call the relation $\equiv_{pos}$ the *positive equivalence* on $\mathfrak{L}$ and denote the equivalence class of a logic $\mathbf{L}$ under $\equiv_{pos}$ by $[\mathbf{L}]_{pos}$.

It should be clear that sets of the form $[\mathbf{L}]_{pos}$ are convex subsets of $\mathfrak{L}$. Hence, the relation $\equiv_{pos}$ induces a system of slices on $\mathfrak{L}$, which we call the *system of positive slices* on $\mathfrak{L}$ and denote by $\mathfrak{L}/{\equiv_{pos}}$; we call elements of $\mathfrak{L}/{\equiv_{pos}}$ *positive slices*. The relation $\leqslant_{pos}$ naturally induces a partial order $\preccurlyeq_{pos}$ on $\mathfrak{L}/{\equiv_{pos}}$: if $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{L}/{\equiv_{pos}}$, then

$$\mathcal{S}_1 \preccurlyeq_{pos} \mathcal{S}_2 \;\rightleftharpoons\; \mathbf{L}_1 \leqslant_{pos} \mathbf{L}_2 \text{ whenever } \mathbf{L}_1 \in \mathcal{S}_1 \text{ and } \mathbf{L}_2 \in \mathcal{S}_2.$$

The relation $\preccurlyeq_{pos}$ is well defined since whether $\mathbf{L}_1 \leqslant_{pos} \mathbf{L}_2$ is independent of the choice of $\mathbf{L}_1 \in \mathcal{S}_1$ and $\mathbf{L}_2 \in \mathcal{S}_2$.

3.3.2 Least logics of positive slices

It should be clear that every positive slice is closed under all (finite and infinite) non-empty intersections, and hence contains a least logic. We next characterize least logics of positive slices:

Proposition 3.2 *For every logic* $\mathbf{L}$*, the following conditions are equivalent:*

(*i*) $\mathbf{L}$ *is the least logic of a positive slice;*

(*ii*) $\mathbf{L} \subseteq \mathbf{QH}+\mathbf{L}^+$*;*

(*iii*) $\mathbf{L} = \mathbf{QH}+\mathbf{L}^+$*;*

(*iv*) $\mathbf{L}$ *is a positively axiomatizable logic.*

Proof (*i*) ⇒ (*ii*): Let $\mathbf{L}$ be the least logic of a positive slice $\mathcal{S}$. By Lemma 3.1, $(\mathbf{QH}+\mathbf{L}^+) \in \mathcal{S}$. Since $\mathbf{L}$ is the least in $\mathcal{S}$, it follows that $\mathbf{L} \subseteq \mathbf{QH}+\mathbf{L}^+$.

(*ii*) ⇒ (*iii*): Since $\mathbf{QH} \subseteq \mathbf{L}$ and $\mathbf{L}^+ \subseteq \mathbf{L}$, surely $\mathbf{QH}+\mathbf{L}^+ \subseteq \mathbf{L}$.

(*iii*) ⇒ (*iv*): Immediate from the definition of positive axiomatizability.

(*iv*) ⇒ (*i*): Suppose that $\mathbf{L} = \mathbf{QH}+\Gamma$, for some $\Gamma \subseteq \mathcal{L}^+$. Then, $\Gamma \subseteq \mathbf{L}^+$. Let $\mathbf{L} \equiv_{pos} \mathbf{L}_0$, i.e., $\mathbf{L}^+ = \mathbf{L}_0^+$. Then, $\Gamma \subseteq \mathbf{L}_0^+ \subseteq \mathbf{L}_0$. Hence, $\mathbf{QH}+\Gamma \subseteq \mathbf{L}_0$, i.e., $\mathbf{L} \subseteq \mathbf{L}_0$. □

Lemma 3.3 *If logics* $\mathbf{L}_1$ *and* $\mathbf{L}_2$ *are positively axiomatizable, then the following conditions are equivalent:*

(*i*) $\mathbf{L}_1 \leqslant_{pos} \mathbf{L}_2$*;*

(*ii*) $\mathbf{L}_1 \subseteq \mathbf{L}_2$.

Proof $(i) \Rightarrow (ii)$: Let $\mathbf{L}_1 \leqslant_{pos} \mathbf{L}_2$, i.e., $\mathbf{L}_1^+ \subseteq \mathbf{L}_2^+$. Then, $\mathbf{QH}+\mathbf{L}_1^+ \subseteq \mathbf{QH}+\mathbf{L}_2^+$. By Proposition 3.2, $\mathbf{L}_1 = \mathbf{QH} + \mathbf{L}_1^+$ and $\mathbf{L}_2 = \mathbf{QH} + \mathbf{L}_2^+$. Thus, $\mathbf{L}_1 \subseteq \mathbf{L}_2$.

$(ii) \Rightarrow (i)$: If $\mathbf{L}_1 \subseteq \mathbf{L}_2$, then $\mathbf{L}_1^+ \subseteq \mathbf{L}_2^+$, i.e., $\mathbf{L}_1 \leqslant_{pos} \mathbf{L}_2$. □

We denote the least logic of a positive slice $\mathcal{S}$ by $\mathbf{L}_{pos}^{\mathcal{S}}$. Proposition 3.2 immediately gives us the following:

Lemma 3.4 *Let $\mathcal{S}$ be a positive slice. Then,*

(1) $\mathbf{L}_{pos}^{\mathcal{S}} = \mathbf{QH} + \mathbf{L}^+$, *for every* $\mathbf{L} \in \mathcal{S}$;

(2) $\mathbf{L}_{pos}^{\mathcal{S}}$ *is the unique positively axiomatizable logic in* $\mathcal{S}$.

Proposition 3.5 *The system $\mathfrak{L}/{\equiv_{pos}}$, partially ordered by $\preccurlyeq_{pos}$, is a complete lattice isomorphic to $\mathfrak{L}_{pos}$.*

Proof By Lemma 3.4 (2), there exists a bijection $f\colon \mathcal{S} \mapsto \mathbf{L}_{pos}^{\mathcal{S}}$ between $\mathfrak{L}/{\equiv_{pos}}$ and $\mathfrak{L}_{pos}$. By definition of $\preccurlyeq_{pos}$,

$$\mathcal{S} \preccurlyeq_{pos} \mathcal{S}' \iff \mathbf{L}_{pos}^{\mathcal{S}} \leqslant_{pos} \mathbf{L}_{pos}^{\mathcal{S}'}.$$

Moreover, by Lemma 3.3,

$$\mathbf{L}_{pos}^{\mathcal{S}} \leqslant_{pos} \mathbf{L}_{pos}^{\mathcal{S}'} \iff \mathbf{L}_{pos}^{\mathcal{S}} \subseteq \mathbf{L}_{pos}^{\mathcal{S}'}.$$

Hence, the lattices $\mathfrak{L}/{\equiv_{pos}}$ and $\mathfrak{L}_{pos}$ are isomorphic. □

Proposition 3.5 shall be relied on in a more detailed study of positive slices that we hope to undertake in the near future.

3.3.3 Maximal logics of positive slices

The logical sum of a family of logics with the same positive fragment might itself have a positive fragment larger than the positive fragment of the summands. Hence, positive slices are not guaranteed to be closed under logical sums, and so are not guaranteed to have largest logics. It should, however, be obvious that, if $\mathcal{C}$ is a chain of logics with the same positive fragment, say P, then $(\bigcup \mathcal{C})^+ = P$. Hence, by Zorn's lemma, every logic of a positive slice $\mathcal{S}$ is included in a logic maximal in $\mathcal{S}$. Thus, every positive slice $\mathcal{S}$ is bounded below by the least logic $\mathbf{L}_{pos}^{\mathcal{S}}$ and above by the antichain $\mathfrak{M}_{pos}^{\mathcal{S}}$ of its maximal logics. In other words, every positive slice $\mathcal{S}$ is representable as follows:

$$\{\mathbf{L} : \exists\, \mathbf{L}' \in \mathfrak{M}_{pos}^{\mathcal{S}}\ (\mathbf{L}_{pos}^{\mathcal{S}} \subseteq \mathbf{L} \subseteq \mathbf{L}')\}.$$

If $\mathbf{L}_0 \in \mathfrak{L}$ and $\mathfrak{A}$ is an antichain of logics such that $\mathbf{L}_0 \subseteq \mathbf{L}$ whenever $\mathbf{L} \in \mathfrak{A}$, then we say that the set $\{\mathbf{L} : \exists\, \mathbf{L}' \in \mathfrak{A}\ (\mathbf{L}_0 \subseteq \mathbf{L} \subseteq \mathbf{L}')\}$ is a *tulip* in $\mathfrak{L}$. Thus, every positive slice $\mathcal{S}$ is a tulip contained between $\mathbf{L}_{pos}^{\mathcal{S}}$ and $\mathfrak{M}_{pos}^{\mathcal{S}}$.

Notice that segments are just tulips whose upper antichains are singletons.

4 Kripke semantics

In this section, we recall Kripke sheaf and Kripke frame semantics for superintuitionistic predicate logics and define some important logics characterized using these types of semantics.

4.1 Kripke sheaves and frames

A *Kripke sheaf* [4, Section 3.6] is a triple $F = \langle W, D, \rho\rangle$ where

- W is a non-empty poset with the partial order $\leqslant$; elements of W are called *points*, or *worlds*;
- $D = \{D_u : u \in W\}$ is a system of non-empty domains;
- $\rho = \{\rho_{uv} : u \leqslant v\}$ is a system of *transition maps* $\rho_{uv} : D_u \to D_v$, subject to the following conditions:
  - · ρ_{uu} is the identity on D_u, for all $u \in W$;
  - · $\rho_{uw} = \rho_{vw} \circ \rho_{uv}$, for all $u, v, w \in W$, i.e.,

$$\forall u, v, w \in W\ [u \leqslant v \leqslant w \Rightarrow \forall a \in D_u\, \rho_{uw}(a) = \rho_{vw}(\rho_{uv}(a))].$$

We say that a Kripke sheaf $F = \langle W, D, \rho\rangle$ is a *Kripke sheaf over the poset* W. If $\boldsymbol{a} = \langle a_1, \ldots, a_n\rangle \in D_u^n$, then $\rho_{uv}(\boldsymbol{a})$ denotes the tuple $\langle \rho_{uv}(a_1), \ldots, \rho_{uv}(a_n)\rangle$.

A *Kripke frame* is a Kripke sheaf satisfying the following conditions:

- *expanding domains condition*: if $u \leqslant v$, then $D_u \subseteq D_v$;
- *the identity condition*: if $u \leqslant v$ and $a \in D_u$, then $\rho_{uv}(a) = a$.

Since in frames transition maps are uniquely determined by the identity condition, to simplify notation, we omit the mention of transition maps when talking about Kripke frames, presenting them simply as pairs $\langle W, D\rangle$. We say that a Kripke frame has a *constant domain* if $D_u = D_v$ whenever $u, v \in W$. The corresponding Kripke sheaves are obtained by requiring all maps ρ_{uv} to be surjective; we are not aware of special category-theoretic terminology for such Kripke sheaves; here we call them *surjective*.

A *valuation* on a sheaf $F = \langle W, D, \rho\rangle$ is a map ζ sending an n-ary predicate letter P and a world $u \in W$ to a subset $\zeta(u, P^n)$ of D_u^n; the map ζ is required to satisfy the following *heredity condition*: for all $u, w \in W$ and $\boldsymbol{a} \in D_u^n$,

$$u \leqslant v \ \&\ \boldsymbol{a} \in \zeta(u, P^n) \implies \rho_{uv}(\boldsymbol{a}) \in \zeta(v, P^n).$$

If F is a Kripke sheaf and ζ a valuation on F, then a tuple $M = \langle F, \zeta\rangle$ is called a *Kripke sheaf model*; if, additionally, F is a Kripke frame, then M is called simply a *Kripke model*. If $u \in W$, then a D_u*-sentence* is an expression obtained from a formula B by substituting (constants corresponding to) elements of D_u for all occurrences of parameters in B. Note that sentences, i.e., formulas without parameters, are just D_u-sentences without any occurrences of constants from D_u. We occasionally write $A(a_1, \ldots, a_n)$ to mean that A is a D_u-sentence containing no constants from D_u beside $a_1, \ldots, a_n$.

Truth of a D_u-sentence $A(\boldsymbol{a})$, with $\boldsymbol{a} \in D_u^n$, for some n, in a Kripke sheaf model M at a point u is defined by recursion (the clauses for $\bot$, $\vee$, and $\wedge$ are as in propositional logic):

- $M, u \Vdash P(\boldsymbol{a})$ if $\boldsymbol{a} \in \zeta(u, P)$;
- $M, u \Vdash (A_1 \to A_2)(\boldsymbol{a})$ if
$$\forall u' \geqslant u\ [M, u' \Vdash A_1(\rho_{uu'}(\boldsymbol{a})) \Rightarrow M, u' \Vdash A_2(\rho_{uu'}(\boldsymbol{a}))];$$

- $M, u \Vdash (\exists x\, A_1)(\boldsymbol{a})$ if $\exists b \in D_u\ [M, u \Vdash A_1(b, \boldsymbol{a})]$;
- $M, u \Vdash (\forall x\, A_1)(\boldsymbol{a})$ if $\forall u' \geqslant u\, \forall b \in D_{u'}\ [M, u' \Vdash A_1(b, \rho_{uu'}(\boldsymbol{a}))]$.

We say that a formula A is

- *true at a point* u of a model M, and write $M, u \Vdash A$, if the D_u-sentence $\bar{\forall} A$ is true in M at u;
- *valid on a Kripke sheaf* F, and write $F \Vdash A$, if $M, u \Vdash A$ holds for every point u of F and every model M over F;
- *valid on a class of Kripke sheaves* if it is valid on every sheaf from the class.

The set of formulas valid on a class $\mathcal{F}$ of Kripke sheaves is denoted by $\mathbf{L}\mathcal{F}$; if $\mathcal{F} = \{F\}$, we write $\mathbf{L}F$ instead of $\mathbf{L}\{F\}$. It is well known that, if F is a Kripke sheaf, then $\mathbf{L}F$ is a logic, called the *logic of* F. Consequently, if $\mathcal{F}$ is a class of Kripke sheaves, then $\mathbf{L}\mathcal{F} = \bigcap\{\mathbf{L}F : F \in \mathcal{F}\}$ is a logic, called the *logic of* $\mathcal{F}$. Logics representable as $\mathbf{L}\mathcal{F}$, for some class $\mathcal{F}$ of Kripke sheaves, are called *Kripke sheaf complete*. Logics representable as $\mathbf{L}\mathcal{F}$, for some class $\mathcal{F}$ of Kripke frames, are called *Kripke complete*.

It is well known [4, Lemma 3.6.20] that the logic of all Kripke sheaves over a class $\mathcal{W}$ of posets coincides with the logic of all Kripke frames over $\mathcal{W}$; hence, we denote this logic by $\mathbf{L}\mathcal{W}$. The same is true for constant domains: the logic of all surjective Kripke sheaves over a class $\mathcal{W}$ of posets coincides with the logic of all Kripke frames with constant domains over $\mathcal{W}$; hence, we denote this logic by $\mathbf{L}_c\mathcal{W}$.

4.2 Subsheaves and rooted sheaves

If W is a poset with a partial order $\leqslant$ and $w_0 \in W$, then a *poset generated by* w_0, denoted by $W{\uparrow}w_0$, is a substructure of W with the set of points $\{w \in W : w_0 \leqslant w\}$. If $F = \langle W, D, \rho\rangle$ is a Kripke sheaf and $w_0 \in W$, then the *subsheaf of* F *generated by* w_0 is the Kripke sheaf $F{\uparrow}w_0 = \langle W{\uparrow}w_0, D', \rho'\rangle$ where D' and ρ' are restrictions to $W{\uparrow}w_0$ of, respectively, D and ρ. A world $w_0 \in W$ is a *root* of F if $F{\uparrow}w_0 = F$. A sheaf is *rooted* if it has a root.

In Section 5.3, we use the following fact [4, Lemma 1.16.3] about formulas of the form δA (see Section 2.2):

Proposition 4.1 *Let* $F = \langle W, R, D\rangle$ *be a Kripke sheaf with root* w_0. *Then,*

$$F \Vdash \delta A \iff \forall w \in W \setminus \{w_0\}\ F{\uparrow}w \Vdash A.$$

4.3 Some families of posets and their logics

For the purposes of this paper, the following classes of posets shall be of interest:

- $\mathcal{W}_{po}$, the class of all posets;
- $\mathcal{W}_g$, the class of all posets with a greatest element;
- $\mathcal{HEI}_h$, where $h < \omega$, the class of posets of height at most h;
- $\mathcal{WID}_n$, where $n < \omega$, the class of posets of width at most n;

- $\mathcal{HEI}_\omega = \bigcup\{\mathcal{HEI}_h : h < \omega\}$, the class of posets of finite height;
- $\mathcal{WID}_\omega = \bigcup\{\mathcal{WID}_n : n < \omega\}$, the class of posets of finite width;
- $\mathcal{CH}_h = \mathcal{WID}_1 \cap \mathcal{HEI}_h$, where $h < \omega$, the class of chains of height at most h;
- $\mathcal{CH}_\omega = \bigcup\{\mathcal{C}_h : h < \omega\} = \mathcal{WID}_1 \cap \mathcal{HEI}_\omega$, the class of all finite chains;
- $\mathcal{FIN} = \mathcal{HEI}_\omega \cap \mathcal{WID}_\omega$, the class of all finite posets.

It should be clear that

$$\begin{array}{llll} \mathbf{L}\mathcal{HEI}_\omega & = \bigcap\limits_{h<\omega} \mathbf{L}\mathcal{HEI}_h; & \mathbf{L}_c\mathcal{HEI}_\omega & = \bigcap\limits_{h<\omega} \mathbf{L}_c\mathcal{HEI}_h; \\ \mathbf{L}\mathcal{WID}_\omega & = \bigcap\limits_{n<\omega} \mathbf{L}\mathcal{WID}_n; & \mathbf{L}_c\mathcal{WID}_\omega & = \bigcap\limits_{n<\omega} \mathbf{L}_c\mathcal{WID}_n. \end{array}$$

We next recall known facts about logics of these classes of posets and about some closely related logics:

(4.1) As shown in [8],

$$\begin{array}{lll} \mathbf{L}\mathcal{W}_{po} & = & \mathbf{QH}; \\ \mathbf{L}_c\mathcal{W}_{po} & = & \mathbf{QH} + CD. \end{array}$$

(4.2) By [3, Theorems 5.8 and 5.12],

$$\begin{array}{lll} \mathbf{L}\mathcal{W}_g & = & \mathbf{QH} + K + J; \\ \mathbf{L}_c\mathcal{W}_g & = & \mathbf{QH} + K + J + CD. \end{array}$$

(4.3) As shown in [26],

$$\mathbf{L}\mathcal{HEI}_h = \mathbf{QH} + P_h^+.$$

(4.4) Even though formulas P_h are propositionally complete ($\mathbf{H} + P_h$ is the propositional logic of $\mathcal{HEI}_h$), if $h > 1$, then the logic $\mathbf{QH} + P_h$ is Kripke, and hence Kripke sheaf, incomplete: H. Ono proved that $\mathbf{QH} + P_h \nvdash K$ whenever $h > 1$; on the other hand, for every such h, the class $\mathcal{HEI}_h$ validates K. The logic $\mathbf{QH} + P_1$ is Kripke complete since $\mathbf{QH} + P_1 = \mathbf{L}\mathcal{HEI}_1 = \mathbf{QC}$ (here, $\mathbf{QC}$ is the classical predicate logic).

(4.5) As follows from [10, Theorem 11],

$$\mathbf{L}_c\mathcal{HEI}_h = \mathbf{QH} + P_h \wedge CD = \mathbf{QH} + P_h^+ \wedge CD.$$

(4.6) As follows from [14, Theorem 3.9], for every $n < \omega$,

$$\mathbf{L}_c\mathcal{WID}_n = \mathbf{QH} + Wid_n \wedge CD.$$

(4.7) As shown in [2,17],

$$\mathbf{L}\mathcal{WID}_1 = \mathbf{L}\mathbb{Q} = \mathbf{L}\mathbb{R} = \mathbf{QH} + Z.$$

(4.8) As shown in [21],

$$\mathbf{L}_c\mathcal{WID}_1 = \mathbf{L}_c\mathbb{Q} = \mathbf{QH} + Z \wedge CD.$$

(4.9) As shown in [2, p. 334],

$$\begin{array}{rcl}\mathbf{L}(\mathcal{WID}_1 \cap \mathcal{W}_g) & = & \mathbf{QH} + K + Z;\\ \mathbf{L}_c(\mathcal{WID}_1 \cap \mathcal{W}_g) & = & \mathbf{QH} + K + Z + CD.\end{array}$$

(4.10) $\mathbf{L}_c\mathbb{R}$ is a finitely axiomatizable proper extension of $\mathbf{L}_c\mathbb{Q}$ [21].

(4.11) $\mathbf{L}\mathcal{HEI}_\omega$ and $\mathbf{L}_c\mathcal{HEI}_\omega$ are Π^0_1-hard [15, Corollary 1.2] and hence not recursively axiomatizable.

(4.12) As shown in [10], $\mathbf{L}\mathcal{CH}_h = \mathbf{QH} + P_h^+ + Z$; on the other hand, if $h > 1$, then $\mathbf{QH} + P_h + Z$ is Kripke sheaf incomplete.

(4.13) $\mathbf{L}_c\mathcal{CH}_h = \mathbf{QH} + P_h^+ + Z + CD = \mathbf{QH} + P_h + Z + CD$.

(4.14) As shown in [15], $\mathbf{L}\mathcal{CH}_\omega$ and $\mathbf{L}_c\mathcal{CH}_\omega$ are both Π^0_1-hard and are both in Π^0_2.

(4.15) $\mathbf{L}\mathcal{FIN}$ and $\mathbf{L}_c\mathcal{FIN}$ are both Π^0_1-hard [15, Corollary 2.1] and are both in Π^0_2.

Remark 4.2 It is not known whether logics $\mathcal{WID}_n$, with $n > 1$, are recursively axiomatizable; we conjecture that the answer is negative; if our conjecture is true, then logics $\mathbf{QH}+Wid_n$ are Kripke incomplete, i.e., are proper sublogics of $\mathbf{L}\mathcal{WID}_n$.[2]

5 Main results

In this section, we present our results on positive slices obtained so far.

5.1 A continuum of degenerate positive slices

5.1.1 Degenerate slices

We say that a positive slice $\mathcal{S}$ is *degenerate* if it is a singleton. By Proposition 3.2, every degenerate slice $\mathcal{S}$ has the form $\{\mathbf{L}^{\mathcal{S}}_{pos}\}$. Moreover, the following is true:

Lemma 5.1 *A slice of a logic* $\mathbf{L}$ *is degenerate if, and only if, the following conditions simultaneously hold:*

(1) $\mathbf{L}$ *is a positively axiomatizable logic;*

(2) *for every logic* $\mathbf{L}_0$*, if* $\mathbf{L} \subset \mathbf{L}_0$*, then* $\mathbf{L}^+ \subset \mathbf{L}_0^+$*.*

Proof Suppose that (1) and (2) hold, and let $\mathcal{S}$ be the positive slice of $\mathbf{L}$. By (1) and Proposition 3.2, $\mathbf{L}$ is the least logic of $\mathcal{S}$. Due to (2), $\mathbf{L}$ is also a maximal logic of $\mathcal{S}$. Hence, $\mathcal{S} = \{\mathbf{L}\}$.

Conversely, suppose that $\mathcal{S}$ is a positive slice such that $\mathcal{S} = \{\mathbf{L}\}$. Then, $\mathbf{L}$ is the least logic of $\mathcal{S}$; hence, by Proposition 3.2, $\mathbf{L}$ is positively axiomatizable, i.e., (1) holds. Since $\mathcal{S}$ contains no logics beside $\mathbf{L}$, no logic other than $\mathbf{L}$ has the same positive fragment as $\mathbf{L}$; hence, (2) holds, as well. □

[2] The third author had established Kripke incompleteness of $\mathbf{QH} + Wid_2 \wedge P_3$; he believes that the Kripke completion of this logic is not recursively enumerable; however, a proof, as well as a proof of incompleteness for $n \geqslant 2$ and $h \geqslant 3$, is likely to be quite complicated.

We say that a logic $\mathbf{L}$ is *hereditarily positively axiomatizable* if both $\mathbf{L}$ and all its proper extensions are positively axiomatizable.

Corollary 5.2 *The slice of a hereditarily positively axiomatizable logic is degenerate.*

Proof Let $\mathbf{L}$ be a hereditarily positively axiomatizable logic. Due to Lemma 5.1, it suffices to show that $\mathbf{L} \subset \mathbf{L}_0$ implies $\mathbf{L}^+ \subset \mathbf{L}_0^+$, for every $\mathbf{L}_0$.

Suppose that $\mathbf{L} \subset \mathbf{L}_0$. Then, $\mathbf{L}^+ \subseteq \mathbf{L}_0^+$ and $\mathbf{L}_0$ is positively axiomatizable. Suppose, for contradiction, that $\mathbf{L}^+ = \mathbf{L}_0^+$. Since $\mathbf{L}$ and $\mathbf{L}_0$ are positively axiomatizable, it follows, by Proposition 3.2, that $\mathbf{L} = \mathbf{QH} + \mathbf{L}^+$ and $\mathbf{L}_0 = \mathbf{QH} + \mathbf{L}_0^+$, and so $\mathbf{L} = \mathbf{L}_0$, contrary to the assumption. Hence, $\mathbf{L}^+ \subset \mathbf{L}_0^+$. □

5.1.2 Superclassical logics

We shall consider the lattice $\mathfrak{L}_{\mathbf{QC}} = \{\mathbf{L} \in \mathcal{L} : \mathbf{QC} \subseteq \mathbf{L}\}$ of *superclassical logics*, i.e., superintuitionistic predicate logics extending the classical predicate logic $\mathbf{QC}$. We begin with the following decreasing chain of all Kripke complete extensions of $\mathbf{QC}$ (notice that all these logics are, indeed, superintuitionistic predicate logics):

- $\mathbf{QC}_m$, where $0 < m < \omega$, is the logic of Kripke frames over singleton posets with m-element domains; thus, $\mathbf{QC}_m$ is the set of formulas classically valid over m-element domains;
- $\mathbf{QC}_\omega = \bigcap_{m=1}^{\infty} \mathbf{QC}_m$ is the logic of Kripke frames over singleton posets with finite domains; thus, $\mathbf{QC}_\omega$ is the set of formulas classically valid over finite domains;
- $\mathbf{QC}_0$ is the inconsistent logic $\mathcal{L}$.[3]

It is well known that both $\mathbf{QC}_0$ and $\mathbf{QC}$ are finitely positively axiomatizable over $\mathbf{QH}$ by, respectively, p and Pierce's law $((p \to q) \to p) \to p$. By Trakhtenbrot's theorem [22], $\mathbf{QC}_\omega$ is Π_1^0-complete, and so is not finitely axiomatizable.

The lattice $\mathfrak{L}_{\mathbf{QC}}$ includes the infinite segment $[\mathbf{QC}, \mathbf{QC}_\omega]$ of logics between $\mathbf{QC}$ and $\mathbf{QC}_\omega$. The following is known [18, Section 0.6]:

Fact 5.3 *The lattice* $\mathfrak{L}_{\mathbf{QC}_\omega} = \{\mathbf{L} \in \mathcal{L} : \mathbf{QC}_\omega \subseteq \mathbf{L}\}$ *of all extensions of* $\mathbf{QC}_\omega$ *is just the decreasing* $(\omega + 1)$*-chain* $\{\mathbf{QC}_m : 0 \leqslant m \leqslant \omega\}$*.*

Fact 5.4 *The lattice* $\mathfrak{L}_{\mathbf{QC}}$ *is just* $\mathfrak{L}_{\mathbf{QC}_\omega} \cup [\mathbf{QC}, \mathbf{QC}_\omega]$*; in other words,* $\mathfrak{L}_{\mathbf{QC}}$ *does not contain logics incomparable with* $\mathbf{QC}_\omega$*.*

Wajsberg [23] had shown that the cardinality of the lattice $\mathfrak{L}_{\mathbf{QC}}$, and hence of the segment $[\mathbf{QC}, \mathbf{QC}_\omega]$, is continuum.

---

[3] By analogy with the case when $m > 0$, one might think of $\mathbf{QC}_0$ as the logic of Kripke frames that are singleton posets with 0-element domains: such Kripke frames do not exist, hence $\mathbf{QC}_0$ is the logic of the empty class of Kripke frames.

5.1.3 A continuum of logics with degenerate slices

To obtain a continuum of logics with degenerate slices, we show that the classical predicate logic **QC** is hereditarily positively axiomatizable:

Proposition 5.5 *Every superclassical logic is positively axiomatizable.*

Proof For every formula A, we choose a nullary letter q not in A and define positive formulas A' and A'' as follows:

$$A' = [q/\bot]A, \qquad A'' = (q \to A') \to A'.$$

We shall prove that A and A'' are deductively equivalent in **QC**.

We start by proving that

$$\mathbf{QH} \vdash (q \vee \neg q) \wedge A \to A''. \tag{5.1}$$

We use reasoning by cases. Case q: It should be clear that $\mathbf{QH} \vdash q \to A''$; hence, $\mathbf{QH} \vdash q \wedge A \to A''$. Case $\neg q$: Since $\mathbf{QH} \vdash \neg q \to (\bot \leftrightarrow q)$, it follows, by induction on A, using the equivalence replacement rule, that $\mathbf{QH} \vdash \neg q \to (A \leftrightarrow A')$. Since $\mathbf{QH} \vdash A' \to ((q \to A') \to A')$, we obtain $\mathbf{QH} \vdash \neg q \to (A \to A'')$. This proves (5.1).

Now, as $\mathbf{QC} \vdash q \vee \neg q$, it follows, by (5.1), that

$$\mathbf{QC} \vdash A \to A''. \tag{5.2}$$

Second, we show that

$$\mathbf{QH} + A'' \vdash A. \tag{5.3}$$

Substituting $\bot$ for q in A'', we obtain $(\bot \to A) \to A$. Since the latter formula is equivalent in **QH** to A, this gives us (5.3).

Thus, A and A'' are deductively equivalent in **QC**, i.e.

$$\mathbf{QC} + A = \mathbf{QC} + A''. \tag{5.4}$$

Now, let **L** be a superclassical logic. By (5.4), $\mathbf{L} + A = \mathbf{L} + A''$, and so

$$\mathbf{L} = \mathbf{QH} + \mathbf{L} = \mathbf{QH} + \{A'' : A \in \mathbf{L}\}. \tag{5.5}$$

Since formulas of the form A'' are positive, (5.5) immediately implies the statement of the proposition. [4] □

The proof of Proposition 5.5 also gives us the following:

Corollary 5.6

(1) *A superclassical logic is finitely positively axiomatizable if, and only if, it is finitely axiomatizable.*

(2) *A superclassical logic is recursively positively axiomatizable if, and only if, it is recursively axiomatizable.*

Since extensions of a superclassical logic are themselves superclassical, Proposition 5.5 immediately gives us the following:

[4] Clearly, in (5.5), i.e., in a positive axiomatization of **L**, it suffices to use only closed formulas from **L**.

Proposition 5.7 *Every superclassical logic is hereditarily positively axiomatizable.*

Finally, by Proposition 5.7 and Corollary 5.2, we obtain the following:

Proposition 5.8 *The positive slice of every superclassical logic is degenerate.*

5.1.4 Finite positive axiomatizations of $\mathbf{QC}_m$

We conclude this section by presenting explicit finite positive axiomatizations for superclassical logics of finite domains, i.e., of logics $\mathbf{QC}_m$, with $0 < m < \omega$. We rely on the following fact, first observed by Skvortsov [18, Section 0.6]:

Fact 5.9 *For every predicate formula A,*

$$\mathbf{QC} + A = \mathbf{QC}_m \iff A \in \mathbf{QC}_m \setminus \mathbf{QC}_{m+1}.$$

Now, for every m with $0 < m < \omega$, we define the formula

$$DOM^*_m = \bigwedge_{i=0}^{m} \exists x\, P_i(x) \to \bigvee_{i \neq j} \exists x\, (P_i(x) \wedge P_j(x)).$$

It is not hard to see that $DOM^*_m \in \mathbf{QC}_m \setminus \mathbf{QC}_{m+1}$, i.e., DOM^*_m is classically valid on domains with m elements, but not on domains with $m+1$ elements. This observation, together with Fact 5.9, immediately gives us the following: [5]

Proposition 5.10 $\mathbf{QC} + DOM^*_m = \mathbf{QC}_m$.

5.2 Non-degenerate slices: main theorem and its corollaries

In this section, we show that positive slices of many well-known Kripke complete and Kripke sheaf complete logics are non-degenerate.

Define the *g-extension* of a poset W, denoted by W^g, to be the poset obtained by adding to W the greatest element; by default, the greatest element of a poset will be denoted by g.

Define the *g-extension* of a Kripke frame $F = \langle W, D \rangle$ by letting

- $D^g_u = D_u$, for every $u \in W$;
- $D^g_g = \bigcup\{D_u : u \in W\}$ (thus, the domain of the greatest point g is the union of all domains from F);
- $F^g = \langle W^g, D^g \rangle$.

Similarly, define the *g°-extension* of a Kripke sheaf $F = \langle W, D, \rho \rangle$ by letting

- $D^\circ_u = D_u$, for every $u \in W$;
- D°_g to be a singleton domain $\{t\}$;
- $\rho^\circ_{ug}(a) = t$, for every $u \in W$ and $a \in D_u$ (thus, t is a common inheritor of all individuals from F);
- $F^\circ = (W^g, D^\circ, \rho^\circ)$.

[5] Axioms DOM^*_m seem to be both simple and natural; we are not, however, aware of their use in the literature.

We note that Kripke sheaf semantics, unlike Kripke frame semantics, allows us to glue together all the individuals of the greatest world, resulting in a compact, convenient, and effective construction.

If $\mathcal{F}$ is a class of Kripke frames, we define $\mathcal{F}^g = \{F^g : F \in \mathcal{F}\}$. We say that a class $\mathcal{F}$ of Kripke frames is *g-closed* if $\mathcal{F}^g \subseteq \mathcal{F}$. Similarly, if $\mathcal{F}$ is a class of Kripke sheaves, we define $\mathcal{F}^\circ = \{F^\circ : F \in \mathcal{F}\}$. We say that a class $\mathcal{F}$ of Kripke sheaves is *g°-closed* if $\mathcal{F}^\circ \subseteq \mathcal{F}$.

We now obtain a sufficient condition for inclusion of positive fragments of Kripke complete and Kripke sheaf complete logics:

Lemma 5.11 (Main lemma)

(1) *If $\mathcal{F}$ is a class of Kripke frames, then $(\mathbf{L}\mathcal{F}^g)^+ \subseteq (\mathbf{L}\mathcal{F})^+$, i.e., $\mathbf{L}\mathcal{F}^g \leqslant_{pos} \mathbf{L}\mathcal{F}$.*

(2) *If $\mathcal{F}$ is a class of Kripke sheaves, then $(\mathbf{L}\mathcal{F}^\circ)^+ \subseteq (\mathbf{L}\mathcal{F})^+$, i.e., $\mathbf{L}\mathcal{F}^\circ \leqslant_{pos} \mathbf{L}\mathcal{F}$.*

Proof (1): Let A be a positive formula not in $(\mathbf{L}\mathcal{F})^+$ (we may assume that A is closed). Then, there exists a Kripke model $M = \langle F, \zeta\rangle$ over a Kripke frame $F = \langle W, D\rangle$ from $\mathcal{F}$ and a world $u_0 \in W$ such that $M, u_0 \not\Vdash A$. Expand the valuation ζ to the valuation ζ^g over the Kripke frame F^g as follows: let $\zeta^g \upharpoonright W = \zeta$ and, for every n-ary predicate letter P, let $\zeta^g(g, P)$ be the set of all n-tuples of elements from D^g_g. Then, all atoms are true at g in M^g. Put $M^g = \langle F^g, \zeta^g\rangle$. It should be clear that M^g satisfies the heredity condition; hence, M^g is a Kripke model. A straightforward induction on D^g_g-sentences shows that

$$M^g, g \Vdash B, \text{ for every positive } D^g_g\text{-sentence } B. \tag{5.6}$$

Now, straightforward induction, using (5.6), shows that, for every $u \in W$ and every positive D_u-sentence B,

$$M, u \Vdash B \iff M^g, u \Vdash B;$$

in other words, the values of positive D_u-sentences are preserved at all non-greatest worlds of M^g. Hence, $M^g, u_0 \not\Vdash A$, and so $A \notin (\mathbf{L}\mathcal{F}^g)^+$.

(2): The argument here is similar. We define a Kripke sheaf model over a Kripke sheaf F° analogously to the definition of the Kripke model M^g from the proof of (1). Since the definition of M^g from (1) did not require to distinguish values of atoms on different individuals from the domain of world g, gluing these individuals together does not affect the truth of formulas: thus, we make all atoms true on the unique individual of the domain of g. The remainder of the argument is essentially identical. □

We next obtain a sufficient condition for the equality of positive fragments of Kripke complete and Kripke sheaf complete logics (this will give us examples of extensive non-degenerate positive slices):

Theorem 5.12 (Main theorem)

(1) *If* $\mathcal{F}$ *is a* g*-closed class of Kripke frames, then* $\mathbf{L}\mathcal{F} \equiv_{pos} \mathbf{L}\mathcal{F}^g$*.*

(2) *If* $\mathcal{F}$ *is a* g°*-closed class of Kripke sheaves, then* $\mathbf{L}\mathcal{F} \equiv_{pos} \mathbf{L}\mathcal{F}^\circ$*.*

Proof (1) Since $\mathcal{F}$ is g-closed, $\mathcal{F}^g \subseteq \mathcal{F}$. Hence $\mathbf{L}\mathcal{F} \subseteq \mathbf{L}\mathcal{F}^g$, and so $(\mathbf{L}\mathcal{F})^+ \subseteq (\mathbf{L}\mathcal{F}^g)^+$, i.e., $\mathbf{L}\mathcal{F} \leqslant_{pos} \mathbf{L}\mathcal{F}^g$. The converse follows by Lemma 5.11.

(2) The argument here is similar to (1). □

Corollary 5.13

(1) $\mathbf{L}\mathcal{W}^g \leqslant_{pos} \mathbf{L}\mathcal{W}$ *and* $\mathbf{L}_c\mathcal{W}^g \leqslant_{pos} \mathbf{L}_c\mathcal{W}$*, for every class* $\mathcal{W}$ *of posets.*

(2) $\mathbf{L}\mathcal{W}^g \equiv_{pos} \mathbf{L}\mathcal{W}$ *and* $\mathbf{L}_c\mathcal{W}^g \equiv_{pos} \mathbf{L}_c\mathcal{W}$*, for every* g*-closed class* $\mathcal{W}$ *of posets.*

Proof Immediate from Lemma 5.11 and Theorem 5.12. □

We next apply Main Theorem to some well-known logics.

Proposition 5.14

(1) $\mathbf{QH} \equiv_{pos} \mathbf{QH} + J + K$;

(2) $\mathbf{QH} + CD \equiv_{pos} \mathbf{QH} + CD + J + K$;

(3) $\mathbf{QH} + Z \equiv_{pos} \mathbf{QH} + Z + K$;

(4) $\mathbf{QH} + CD + Z \equiv_{pos} \mathbf{QH} + CD + Z + K$.

Proof We use facts from Section 4.3.

(1): Since the class $\mathcal{W}_{po}$ of all posets is g-closed, it follows, by Corollary 5.13 (2), that $\mathbf{L}\mathcal{W}_{po} \equiv_{pos} \mathbf{L}\mathcal{W}^g_{po}$. By (4.1), $\mathbf{L}\mathcal{W}_{po} = \mathbf{QH}$. Since $\mathcal{W}^g_{po} = \mathcal{W}_g$, it follows that $\mathbf{L}\mathcal{W}^g_{po} \vdash J \wedge K$. Hence, $\mathbf{QH} \equiv_{pos} \mathbf{QH} + J + K$.

(2)–(4): The argumentation here is similar; use (4.1), (4.7), and (4.8). □

The scope of Proposition 5.14 will become clearer if the reader consults Fact 2.2 (3), Lemma 2.3, and Remark 2.5.

Due to (4.2), and (4.9), the results of Proposition 5.14 are the maximal ones that Theorem 5.12 (1) enables us to obtain. However, using Kripke sheaf semantics, we shall next obtain stronger (see Remark 5.21) results.

Lemma 5.15 *Let* F *be a Kripke sheaf. Then,* $F^\circ \Vdash J \wedge \neg\neg U$*.*

Proof First, $F^\circ \Vdash J$ since F° is a sheaf over a poset with the greatest element g. Second, $F^\circ \Vdash \neg\neg U$ since the domain of g is a singleton and hence $F^\circ \uparrow g \Vdash U$. □

Proposition 5.16 *Let* $\mathcal{C}$ *be one of the following classes of posets:* $\mathcal{W}_{po}$*,* $\mathcal{WID}_n$*, for some* n *such that* $1 < n < \omega$*,* $\mathcal{WID}_\omega$*,* $\mathcal{HEI}_\omega$*, and* $\mathcal{FIN}$*; let also* $\mathbf{L} \in \{\mathbf{L}\mathcal{C}, \mathbf{L}_c\mathcal{C}\}$*. Then,* $\mathbf{L} \equiv_{pos} \mathbf{L} + J + \neg\neg U$*.*

Proof By Lemma 5.15, formulas J and $\neg\neg U$ are valid on every Kripke sheaf of the form F°. Every class of posets mentioned in the proposition is g-closed; thus, the corresponding classes of Kripke sheaves and of surjective Kripke sheaves are g°-closed. Hence, the statement follows by Theorem 5.12 (2). □

Proposition 5.17 *If* $\mathbf{L} \in \{\mathbf{QH} + Z, \mathbf{QH} + Z + CD\}$*, then* $\mathbf{L} \equiv_{pos} \mathbf{L} + \neg\neg U$.

Proof Similar to the proof of Proposition 5.16 (recall that, by Fact 2.2 (3), $\mathbf{QH} + Z \vdash J$). □

Remark 5.18 Proposition 5.17 can be transferred to the logics of the class $\mathcal{CH}_\omega$ of all finite chains.

Remark 5.19 Notice that $\neg\neg U'$, which, by Fact 2.2 (1), is equivalent in $\mathbf{QH}$ to $\neg\neg U$, implies both K and E: namely, in the presence of $\neg\neg U'$, i.e., in the logic $\mathbf{L} = \mathbf{QH} + \neg\neg U = \mathbf{QH} + \neg\neg U'$, we obtain

$$\forall x \neg\neg P(x) \Rightarrow_{\mathbf{L}} \exists x \neg\neg P(x) \Rightarrow_{\mathbf{L}} \neg\neg\exists x\, P(x) \Rightarrow_{\mathbf{L}} \neg\neg\forall x P(x)$$

and

$$\neg\neg\exists x P(x) \Rightarrow_{\mathbf{L}} \neg\neg\forall x P(x) \Rightarrow_{\mathbf{L}} \forall x \neg\neg P(x) \Rightarrow_{\mathbf{L}} \exists x \neg\neg P(x).$$

We next show that Propositions 5.16 and 5.17 are stronger than Proposition 5.14. To that end, we need the following lemma:

Lemma 5.20 $\mathbf{QH} + Z + P_2^+ \nvdash E$.

Proof Recall from Section 4.3 that formulas Z and P_2^+ are valid, respectively, on chains and on posets of height at most 2. Define a Kripke model $M = \langle W, D, \zeta \rangle$ so that $W = \{u, v\}$ is a two-element poset where $u < v$, $D(u) = \{a\}$, $D(v) = \{a, b\}$, $\zeta(u, P) = \varnothing$, and $\zeta(v, P) = \{b\}$. Since W is a chain of height 2, surely $\langle W, D \rangle \Vdash \{J, P_2^+\}$. On the other hand, it is straightforward to check that $M, u \nVdash E$. □

Remark 5.21 To compare Propositions 5.16 and 5.14, observe that, since $\mathbf{QH} + \neg\neg U \vdash E$, Proposition 5.16 implies that

$$\mathbf{QH} \equiv_{pos} \mathbf{QH} + E. \tag{5.7}$$

On the other hand, Proposition 5.14 implies that $\mathbf{QH} \equiv_{pos} \mathbf{L}$ only if $\mathbf{L} \subseteq \mathbf{QH} + J + K$; this statement is weaker than (5.7) since, as we next show, E does not belong to the latter logic: indeed, since $\mathbf{QH} + Z \vdash J$ and $\mathbf{QH} + P_2^+ \vdash K$, it follows, by Lemma 5.20, that $\mathbf{QH} + J + K \nvdash E$.

Similar observations about logics with the axiom Z apply to Propositions 5.17 and 5.14.

Proposition 5.22 $\mathbf{L}_c\mathbb{R} \equiv_{pos} \mathbf{L}_c\mathbb{R} + \neg\neg U$.

Proof We show that $\mathbf{L}_c\mathbb{R} + \neg\neg U \leqslant_{pos} \mathbf{L}_c\mathbb{R}$, i.e., that

$$(\mathbf{L}_c\mathbb{R} + \neg\neg U)^+ \subseteq (\mathbf{L}_c\mathbb{R})^+. \tag{5.8}$$

Denote by $\mathcal{F}$ and $\mathcal{F}^*$ the classes of surjective Kripke sheaves over, respectively, $\mathbb{R}$ and $\mathbb{R}^g$, ordered by the usual $\leqslant$ relation. Then, $\mathbf{L}\mathcal{F} = \mathbf{L}_c\mathbb{R}$ and

$\mathbf{L}\mathcal{F}^* = \mathbf{L}_c\mathbb{R}^g$. Takano proved [21, Theorem (2°) and Proposition 5.1 (2°)] that $\mathbf{L}_c\mathbb{R}^g = \mathbf{L}_c\mathbb{R} + K$.[6] Hence, $\mathbf{L}_c\mathbb{R} \subseteq \mathbf{L}_c\mathbb{R}^g$.[7]

Now, let $\mathcal{F}^\circ = \{F^\circ : F \in \mathcal{F}\}$. Since every sheaf of the form F°, with $F \in \mathcal{F}$, is obviously surjective, surely $\mathcal{F}^\circ \subseteq \mathcal{F}^*$. Hence, $\mathbf{L}\mathcal{F}^* \subseteq \mathbf{L}\mathcal{F}^\circ$, and so

$$\mathbf{L}_c\mathbb{R} \subseteq \mathbf{L}_c\mathbb{R}^g = \mathbf{L}\mathcal{F}^* \subseteq \mathbf{L}\mathcal{F}^\circ.$$

By Lemma 5.15, $\neg\neg U \in \mathbf{L}\mathcal{F}^\circ$. Hence,

$$\mathbf{L}_c\mathbb{R} + \neg\neg U \subseteq \mathbf{L}\mathcal{F}^\circ. \tag{5.9}$$

By Lemma 5.11 (2), $(\mathbf{L}\mathcal{F}^\circ)^+ \subseteq (\mathbf{L}\mathcal{F})^+$, i.e.

$$(\mathbf{L}\mathcal{F}^\circ)^+ \subseteq (\mathbf{L}_c\mathbb{R})^+. \tag{5.10}$$

Lastly, (5.9) and (5.10) immediately imply (5.8). □

We note that Proposition 5.22 can be obtained from Main Theorem since the g-closure of $\mathbb{R}$ validates Takano's axioms for $\mathbf{L}_c\mathbb{R}$ [21, Proposition 5.2]; we, however, believe that our proof of Proposition 5.22 is simpler and more immediate than an appeal to Main Theorem.

5.3 Some restrictions on the application of the main theorem

In this section, we give examples of some interesting classes of posets to which Theorem 5.12 does not apply; the study of their positive fragments shall require techniques that differ from those used here.

Since no class $\mathcal{HEI}_h$, where $h < \omega$, is g-closed, the following is not unexpected (recall from (4.3), (4.4), and (4.5) that $\mathbf{QH} + P_h \subset \mathbf{QH} + P_h^+ = \mathbf{L}\mathcal{HEI}_h$ and $\mathbf{QH} + CD + P_h = \mathbf{QH} + CD + P_h^+ = \mathbf{L}_c\mathcal{HEI}_h$, for every h with $1 < h < \omega$):

Proposition 5.23 *Let* $\mathbf{L}$ *be one of the logics* $\mathbf{QH} + P_h$, $\mathbf{QH} + P_h^+$, *or* $\mathbf{QH} + CD + P_h$, *where* $1 < h < \omega$. *Then,* $\mathbf{L} \not\equiv_{pos} \mathbf{L} + J$.

Proof We first consider the case when $h = 2$. Denote by W_3 a rooted 3-element poset with two maximal elements ('fork'); this is a tree of height 2. It should be clear that $\mathbf{L}_c\mathcal{HEI}_2 \subseteq \mathbf{L}_cW_3$, but $W_3 \not\Vdash Z$; hence, $\mathbf{L}_c\mathcal{HEI}_2 \nvdash Z$. On the other hand, by Lemma 2.6, $\mathbf{QH} + J + P_2 \vdash Z$. Hence, $Z \in (\mathbf{L} + J) \setminus \mathbf{L}$, and so $\mathbf{L} \not\equiv_{pos} \mathbf{L} + J$, for every logic $\mathbf{L}$ from the segment $[\mathbf{QH} + P_2, \mathbf{L}_c\mathcal{HEI}_2]$.

The case when $h > 2$ is similar, using the δ-operation on formulas (see Section 2.2). Recall that that $P_h = \delta^{h-2}P_2$. Due to Lemma 2.6, $\mathbf{QH}+J \vdash P_2 \to Z$. Hence, by Fact 2.1 (2), $\mathbf{QH}+J \vdash \delta^{h-2}P_2 \to \delta^{h-2}Z$, i.e., $\mathbf{QH}+J \vdash P_h \to \delta^{h-2}Z$. On the other hand, since, as we have seen, $\mathbf{L}_c\mathcal{HEI}_2 \nvdash Z$, it follows, by Proposition 4.1, that $\mathbf{L}_c\mathcal{HEI}_h \nvdash \delta^{h-2}Z$. Therefore, $\delta^{h-2}Z \in (\mathbf{L} + J) \setminus \mathbf{L}$, and so $\mathbf{L} \not\equiv_{pos} (\mathbf{L} + J)$, for every logic $\mathbf{L}$ from the segment $[\mathbf{QH} + P_h, \mathbf{L}_c\mathcal{HEI}_h]$. □

---

[6] Even though Takano characterized these logics using Kripke frames, the same logics can, as we have seen at the end of Section 4.1, be characterized using surjective Kripke sheaves.

[7] By the way, this inclusion, besides Takano's completeness results, also follows from the existence of a p-morphism of $\mathbb{R}$ onto $\mathbb{R}^g$; for information on p-morphisms of Kripke frames, consult [4, Section 3.3].

Remark 5.24 Due to Lemma 2.6, $\mathbf{QH} \vdash J \wedge P_2 \to Z$. Hence, by Fact 2.1 (2) and Fact 2.1 (3), $\mathbf{QH} \vdash \delta^{h-2} J \wedge \delta^{h-2} P_2 \to \delta^{h-2} Z$, for every $h > 2$.

Hence, if $h > 2$, then, for every logic $\mathbf{L}$ from Proposition 5.23 parameterized by the said h,

$$\mathbf{L} \not\equiv_{pos} (\mathbf{L} + \delta^{h-2} J).$$

We note that, by Fact 2.1 (1), $\mathbf{L} + \delta^{h-2} J \subset \mathbf{L} + J$, for such logics $\mathbf{L}$.

Proposition 5.25 *Let* $\mathbf{L}$ *be one of the following logics:*

- *an extension of* $\mathbf{QH} + P_2$ *not containing* Z*;*
- $\mathbf{QH} + P_h + Wid_n$ *and* $\mathbf{QH} + P_h^+ + Wid_n$*, with* $1 < h < \omega$ *and* $1 < n < \omega$*;*
- $\mathbf{QH} + CD + P_h + Wid_n$ *(i.e.,* $\mathbf{QH} + CD + P_h^+ + Wid_n$*), with* $1 < h < \omega$ *and* $1 < n < \omega$*.*

Then $\mathbf{L} \not\equiv_{pos} \mathbf{L} + J$*.*

Proof Similar to the proof of Proposition 5.23. □

Remark 5.26 Observe that logics mentioned in the first item of Proposition 5.25 include the logics $\mathbf{QH} + P_2^+ + Wid_n$ of n-branching trees of height 2, as well as Kripke incomplete logics $\mathbf{QH} + P_2 + Wid_n$, with $1 < n < \omega$. The logics of constant domains $\mathbf{QH} + CD + P_2 + Wid_n$ (i.e., $\mathbf{QH} + CD + P_2^+ + Wid_n$) are included, as well.

5.4 On logics that are not positively axiomatizable

Our results have immediate corollaries concerning lack of positive axiomatizability for classes of logics. We give only one example (it is not hard to extend it to similar cases):

Proposition 5.27 *Neither* $\mathbf{QH} + J$*, nor any logic in the interval* $(\mathbf{QH}, \mathbf{QH} + J)$ *is positively axiomatizable.*

Proof Immediate from Proposition 3.2 and Proposition 5.14 (1). □

We next give an example independent from Proposition 3.2 (once again, it is not hard to produce similar examples). Recall that we denote by W_3 a 3-element rooted poset with two maximal elements ('fork'); we also denote by W_4 a 4-element poset where a root sees a two-element anti-chain whose elements see the greatest element ('rhombus').

Proposition 5.28 *No logic in the segment* $[\mathbf{L}W_4, \mathbf{L}_c W_4]$ *is positively axiomatizable.*

Proof Clearly, $W_4 = W_3^g$. Hence, by Corollary 5.13 (1), $\mathbf{L}_c W_4 \leqslant_{pos} \mathbf{L}_c W_3$, i.e., $(\mathbf{L}_c W_4)^+ \subseteq (\mathbf{L}_c W_3)^+$. On the other hand, $J \in \mathbf{L}_c W_4 \setminus \mathbf{L}_c W_3$. Hence, if $\mathbf{L} \in [\mathbf{L}W_4, \mathbf{L}_c W_4]$, then $J \in \mathbf{L} \setminus (\mathbf{QH} + \mathbf{L}^+)$, which implies the statement of the proposition. □

We do not know how to explicitly axiomatize the least logic in the positive slice of $\mathbf{L}W_4$. Logics $\mathbf{L}W_3$ and $\mathbf{L}_c W_3$, as well as logics mentioned in Remark 5.26, are positively axiomatizable.

Remark 5.29 It is well known that propositional extensions of the logic $\mathbf{H}+Z$, i.e., propositional logics of (finite) chains, as well as propositional extensions of $\mathbf{H}+P_2$, i.e., propositional logics of (finite) trees of height at most 2 (and so, in particular, proper extensions of the propositional logic of W_4, i.e., the logic of rhombus), are all finitely positively axiomatizable. It remains unknown if the analogous facts hold for predicate logics.

Problem 5.30 Are $\mathbf{QH} + CD + Z$ (the logic of all chains with constant domains) and $\mathbf{QH} + CD + P_2^+$ (the logic of all trees of height at most 2 with constant domains) hereditarily positively axiomatizable?

Note that the class of extensions of $\mathbf{QH} + CD + Z$ includes the infinite family of the predicate logics of ordinals. The constant domain logics of ordinals have been studied by Minari, Takano, and Ono [9]. It is known [15] that all these logics except logics of finite chains, as well as their expanding domains counterparts, are Π^1_1-hard, and so are not arithmetical.

6 Directions for future work

The present paper is but a first sketch of the study of the system of positive slices in the lattice of superintuitionistic predicate logics.

As already mentioned, our Main Theorem generalises an observation made by Yankov [25] about superintuitionistic propositional logics. In fact, Yankov proved that the propositional logic $\mathbf{H}+J$ of the weak law of the excluded middle is the greatest propositional logic whose positive fragment coincides with that of the intuitionistic propositional logic $\mathbf{H}$ (i.e., in our terminology, the positive slice of $\mathbf{H}$ is the segment $[\mathbf{H}, \mathbf{H} + J]$). Our Main Theorem does not imply an analogous statement for predicate logics. Moreover, we do not know if any of the non-degenerate positive slices of predicate logics are segments. These are questions for future study.

Acknowledgements

We are grateful to anonymous reviewers for helping to improve the presentation of the paper. The first author has been supported by the HSE Academic Fund Program, Project 23-00-022.

References

[1] Chagrov, A. and M. Zakharyaschev, "Modal Logic," Oxford University Press, 1997.

[2] Corsi, G., *Completeness theorem for Dummett's LC quantified and some of its extensions*, Studia Logica **51** (1992), pp. 317–335.

[3] Corsi, G. and S. Ghilardi, *Directed frames*, Archive for Mathematical Logic **29** (1989), pp. 53–67.

[4] Gabbay, D., V. Shehtman and D. Skvortsov, "Quantification in Nonclassical Logic, Volume 1," Studies in Logic and the Foundations of Mathematics **153**, Elsevier, 2009.

[5] Ghilardi, S., *Incompleteness results in Kripke semantics*, Journal of Symbolic Logic **56** (1991), pp. 517–538.

[6] Hosoi, T., *On intermediate logics, I*, Journal of the Faculty of Science of the University of Tokyo, Section I **14** (1967), pp. 293–312.

[7] Hosoi, T., *On intermediate logics, II*, Journal of the Faculty of Science of the University of Tokyo, Section I **16** (1969), pp. 1–12.

[8] Kripke, S., *Semantical analysis of intuitionistic logic I*, in: J. Crossley and M. Dummett, editors, *Formal Systems and Recursive Functions*, Studies in Logic and the Foundations of Mathematics **40**, Elsevier, 1965 pp. 92–130.

[9] Minari, P., M. Takano and H Ono. *Intermediate Predicate Logics Determined by Ordinals*. Journal of Symbolic Logic **55**, pp. 1099–1124

[10] Ono, H., *On finite linear intermediate predicate logics*, Studia Logica **47** (1988), pp. 391–399.

[11] Rybakov, M. and D. Shkatov, *Undecidability of first-order modal and intuitionistic logics with two variables and one monadic predicate letter*, Studia Logica **107** (2019), pp. 695–717. Corrected version available at `https://arxiv.org/abs/1706.05060`.

[12] Rybakov, M. and D. Shkatov, *Algorithmic properties of first-order superintuitionistic logics of finite Kripke frames in restricted languages*, Journal of Logic and Computation **31** (2021), pp. 494–522.

[13] Rybakov, M. and D. Shkatov, *Variations on the Kripke trick*, To appear in Studia Logica, `https://doi.org/10.1007/s11225-023-10093-y`.

[14] Shimura, T., *Kripke completeness of some intermediate predicate logics with the axiom of constant domain and a variant of canonical formulas*, Studia Logica **52** (1993), pp. 23–40.

[15] Skvortsov, D., *On non-axiomatizability of superintuitionistic predicate logics of some classes of well-founded and dually well-founded Kripke frames*, Journal of Logic and Computation **16** (2006), pp. 685–695.

[16] Skvortsov, D., *A remark on superintuitionistic predicate logics of Kripke frames with constant and with nested domains*, Journal of Logic and Computation **21** (2009), pp. 697–713.

[17] Skvortsov, D., *On an axiomatization of superintuitionistic predicate logic of Kripke frames with nested domains over the set of reals*, Topology, Algerbra and Categories in Logic, TACL 2009, ILLC, Amsterdam, 2009.

[18] Skvortsov, D., *On finite domains based slices in the structure of superintuitionistic predicate logics, preview*, Logical Investigations **29** (2023), pp. 101–113.

[19] Skvortsov, D., *On systems of slices in the structure of superintuitionistic predicate* (*or propositional*) *logics I*, Proceedings of the 20th Russian Conference on Artificial Intelligence, Vol. 1, Moscow, 2022, pp. 311–321 (in Russian).

[20] Skvortsov, D., *On systems of slices in the structure of superintuitionistic predicate* (*or propositional*) *logics I*, Pattern Recognition and Image Analysis **33** (2023), pp. 511–516 (English translation of [19]).

[21] Takano, M., *Ordered sets R and Q as bases of Kripke models*, Studia Logica (1987), pp. 137–148.

[22] Trakhtenbrot, B. A., *Impossibility of an algorithm for the decision problem in finite classes*, American Mathematica Society Translations **23** (1963), pp. 1–5.

[23] Wajsberg, M., *Beitrag zur Metamathematik*, Mathematische Annalen **109**, pp. 200- 229. (English translation: *A contribution to metamathematics*, in: M. Wajsberg, "Logical Works", edited by S. J. Surma, Ossolineum 1977, 62-88.)

[24] Wolter, F. and M. Zakharyaschev, *Modal decision problems*, in: P. Blackburn, J. V. Benthem and F. Wolter, editors, *Handbook of Modal Logic*, Elsevier, 2007 pp. 427–489.

[25] Yankov, V. A., *The calculus of the weak "Law of excluded middle"*, Mathematics of the USSR–Izvestiya **32** (1968), pp. 1044-1051.

[26] Yokota, S., *Axiomatization of the first-order intermediate logics of bounded Kripkean heights I*, Mathematical Logic Quarterly **35** (1989), pp. 415–421.