Products of Horn Modal Logics

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Abstract

We study products of unimodal logics characterized by classes of Kripke frames defined by universal Horn formulas, classifying them with respect to the finite model property (FMP). Further, we show that products of modal logics defined only with variable-free axioms have the FMP. We also provide a partial result regarding products of logics with both Horn and variable-free axioms.

Keywords: Horn formula, product of modal logics, filtration

1 Introduction

A *Horn modal logic* is a modal logic characterized by the class of all Kripke frames satisfying several first-order *universal Horn clauses*. Unimodal Horn logics naturally fall into 4 types (cf. [7]); we refer to them as *transitive* (e.g., **K4**, **S4**), *reflexive-symmetric* ($\mathbf{K}, \mathbf{KB}$, **T**, **KTB**), *strong* (**K5**, **S5**), and *uniform* ($\mathbf{K} + \Diamond\Box p \to \Box^2 p$). Logics of the first two types are PSPACE-complete, and those of the other two have the polynomial model property and are coNP-complete [5] [7].

Some bimodal Horn logics are undecidable [7]. Apparently, there is no known decidability criterion for these.

The finite model property (FMP) is known for certain products of Horn logics, including $(\mathbf{K} + \Box p \to \Box^m p) \times \mathbf{S5}$ [3] and $(\mathbf{K} + \Box p \to \Box^m p) \times \mathbf{K}_m$ [8]. However, some other products (e.g., $\mathbf{K4} \times \mathbf{K4}$) are undecidable [4].

In this paper we classify all products of unimodal Horn logics with respect to the FMP. We deduce from [4] that a product of two *transitive* Horn logics is undecidable and does not have the FMP. By employing the *filtration via bisimulation* technique of [8], we establish that all other products of unimodal Horn logics have the FMP.

We also extend this result to products of Horn logics with additional variable-free axioms, provided the two Horn logics are not *uniform*. This includes the following special case: if λ_1 and λ_2 are variable-free, then $(\mathbf{K} + \lambda_1) \times (\mathbf{K} + \lambda_2)$ has the FMP and is decidable.

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2 Preliminaries

Basics. We consider the basic unimodal (**ML**) and bimodal ($\mathbf{ML}_2$) propositional languages; only normal logics are considered. We use the standard Kripke semantics; unimodal (bimodal) Kripke frame are called *frames (2-frames)* for short; the logic characterized by a class of (2-)frames $\mathcal{C}$ is denoted $\operatorname{Log}\mathcal{C}$. First-order formulas with a single binary predicate R are interpreted over frames; the logic characterized by the class of all frames modeling a first-order theory Γ is denoted $\mathbf{K}(\Gamma)$.

Products. The *product* of frames (W_1, R_1) and (W_2, R_2) is the 2-frame $(W_1 \times W_2, R_1', R_2')$, where: $R_1' := \{((x, z), (y, z)) : (x, y) \in R_1, z \in W_2\}$ and $R_2' := \{((z, x), (z, y)) : z \in W_1, (x, y) \in R_2\}$. The *product* of classes of frames $\mathcal{C}_1$ and $\mathcal{C}_2$ is the class of 2-frames $\mathcal{C}_1 \times \mathcal{C}_2 := \{\mathfrak{F}_1 \times \mathfrak{F}_2 : \mathfrak{F}_i \in \mathcal{C}_i\}$. The *product* of logics L_1 and L_2 is the bimodal logic $L_1 \times L_2 := \operatorname{Log}(\operatorname{Fr} L_1 \times \operatorname{Fr} L_2)$, where $\operatorname{Fr} L_i$ is the class of all frames for L_i.

The *commutator* of L_1 and L_2, denoted $[L_1, L_2]$, is the minimal bimodal logic containing the axioms $\Diamond_1\Box_2 p \to \Box_2\Diamond_1 p$ and $\Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p$ and extending $[\Box_1/\Box]L_1 \cup [\Box_2/\Box]L_2$.

Filtration. For a subformula-closed set $\Sigma \subseteq \mathbf{ML}$, a Σ-*filtration* of a Kripke model $(W, R, \mathfrak{V})$ is a Kripke model of the form $(W/{\sim}, S, \mathfrak{V}^{\sim})$ satisfying the following conditions:

(1) if $x \sim y$ and $\varphi \in \Sigma$, then $(\mathfrak{M}, x) \models \varphi \iff (\mathfrak{M}, y) \models \varphi$,
(2) $S \supseteq R^{\sim}$, where $R^{\sim} := \{([x], [y]) : xRy\}$,
(3) if $[x]S[y]$, $(\mathfrak{M}, x) \models \Box\varphi$, and $\Box\varphi \in \Sigma$, then $(\mathfrak{M}, y) \models \varphi$, and
(4) $\mathfrak{V}^{\sim}(p) := [\mathfrak{V}(p)]$.

A filtration satisfying $S = R^{\sim}$ is called a *minimal filtration.* A logic L *admits filtration with respect to a frame* $\mathfrak{F}$ if for any valuation $\mathfrak{V}$ on $\mathfrak{F}$ and any finite subformula-closed set $\Sigma \subseteq \mathbf{ML}$ there exists a finite Σ-filtration of $(\mathfrak{F}, \mathfrak{V})$ based on a frame for L. A logic L *admits fitlration* if it admits filtration with respect to each frame for L. Similar definitions apply to bimodal logics and Kripke models.

Trees. A frame (W, R) is a *tree* with a root $w \in W$ if for any $u \in W$ there exists a unique R-path from w to u.

Pseudo-finitness. A frame (W, R) is s*-pseudo-finite* if there exists an equivalence relation $\sim$ on W such that $|W/{\sim}| \leq s$ and $\sim \circ R \circ \sim = R$. A frame is *pseudo-finite* if it is s-pseudo-finite for some s.

3 Horn Logics

Definition 3.1 A *Horn clause* is a first-order sentence of the following form:

$$\forall \overrightarrow{x} \left(\bigwedge_{s=1}^{m} x_{i_s} R x_{j_s} \to x_{i_0} R x_{j_0} \right).$$

A *Horn theory* is a set of Horn clauses. For a Horn theory Γ, the *Horn* Γ*-closure* of a frame $\mathfrak{F} = (W, R)$ is the frame $\mathfrak{F}^{\Gamma} := (W, R^{\Gamma})$, where R^{Γ} is the minimal relation containing R and satisfying $(W, R^\Gamma) \models \Gamma$. A *Horn logic* is a logic of the form $\mathbf{K}(\Gamma)$, where Γ is a Horn theory.

Definition 3.2 A *tree-clause* corresponding to a finite tree $(W, \widetilde{R})$ and nodes $u_0, v_0 \in W$ is the Horn clause $\forall \overrightarrow{x} \left(\bigwedge_{(u,v)\in\widetilde{R}} x_u R x_v \to x_{u_0} R x_{v_0} \right)$. Its *type* is the pair $(n, m) \in (\mathbb{Z}_{\geq 0})^2$ such that $w\widetilde{R}^n u_0$ and $w\widetilde{R}^m v_0$, where w is the least common ancestor of u_0 and v_0. A *tree-theory* is a set of tree-clauses.

Example 3.3 For any $n, m \geq 0$, the logic $\mathbf{K} + \Diamond^n \Box p \to \Box^m p$ is Horn, as its only axiom is a Sahlqvist modal equivalent of $\forall x, y, z\, (xR^n y \wedge xR^m z \to yRz)$. The latter is a tree-clause of type (n, m).

As shown in [6], tree-clauses have Sahlqvist modal equivalents and are the only (up to equivalence) Horn clauses having modal equivalents at all.

Proposition 3.4 *Every Horn logic coincides with* $\mathbf{K}(\Gamma)$ *for some (possibly infinite) tree-theory* Γ.

Lemma 3.5 ([3], [1]) *Let* Γ_1, Γ_2 *be Horn theories; set* $L_i := \mathbf{K}(\Gamma_i)$. *If* $\varphi \notin [L_1, L_2]$, *then there exist trees* $\mathfrak{T}_i$ *with roots* w_i *such that* $\left(\mathfrak{T}_1^{\Gamma_1} \times \mathfrak{T}_2^{\Gamma_2}, (w_1, w_2)\right) \not\models \varphi$. *In particular,* $[L_1, L_2] = L_1 \times L_2$.

Remark 3.6 It follows from Lemma 3.5 that the product $\mathbf{K}(\Gamma_1) \times \mathbf{K}(\Gamma_2)$ coincides with $\mathrm{Log}\,\{\mathfrak{F}_1 \times \mathfrak{F}_2 \,:\, \mathfrak{F}_1 \models \Gamma_1, \mathfrak{F}_2 \models \Gamma_2\}$.

Remark 3.7 Proposition 3.4 and Lemma 3.5 also hold for polymodal Horn logics, with "n-trees" substituted for trees; cf. [3] [6].

4 Classification of Unimodal Horn Logics

We divide unimodal Horn logics into 4 classes, similarly to the classification used in [7].

Definition 4.1 A tree-theory Γ is —

- *reflexive-symmetric* if the type of each clause in Γ is $(0, 0)$, $(1, 0)$, or $(0, 1)$;
- *transitive* if all clauses in Γ have types of the form $(0, m)$, and at least one of them is with $m > 1$;
- *uniform* if all clauses in Γ have types of the form $(n, n + 1)$, and at least one of them is with $n > 0$;
- *strong* in all other cases.

Lemma 4.2 *If* Γ *is a strong tree-theory, then there exist integers* $d, s \geq 0$ *such that for any frame* $(W, R) \models \Gamma$ *and any* $u \in W$ *satisfying* $R^{-d}(u) \neq \emptyset$ *the restriction of* R *to* $\left\{x \in R^{<\infty}(u) \,:\, R^d(x) \neq \emptyset\right\}$ *is* s*-pseudo-finite.*

We will need the following properties of logics of specific types:

Lemma 4.3 *Any reflexive-symmetric, transitive, or strong tree-theory is equivalent to some finite subtheory.*

Lemma 4.4 *If* Γ *is a transitive tree-theory, then* $\mathbf{K}(\Gamma)$ *admits filtration.*

Lemma 4.4 is proven by appropriately generalizing the proof of FMP for $\mathbf{K} + \Box p \to \Box^m p$ from [2].

5 Products without the FMP

Our main result is as follows.

Theorem 5.1 *Let Γ_1 and Γ_2 be tree-theories.*

1) If both Γ_1 and Γ_2 are transitive, then $\mathbf{K}(\Gamma_1) \times \mathbf{K}(\Gamma_2)$ is undecidable and does not have the fmp.

2) If Γ_1 (or Γ_2) is not transitive, then $\mathbf{K}(\Gamma_1) \times \mathbf{K}(\Gamma_2)$ has the fmp.[2]

First, we derive the negative part of the claim from the following fact.

Lemma 5.2 ([4]) *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of transitive frames both containing frames of infinite depth. Then $\mathrm{Log}(\mathcal{C}_1 \times \mathcal{C}_2)$ is undecidable.*

Proof of Theorem 5.1(1) For $i \in \{1, 2\}$ choose an integer $l_i > 0$ such that $\mathcal{C}_i := \{(W, R^{l_i}) : (W, R) \models \Gamma_i\}$ contains only transitive frames. Observe that Lemma 5.2 is applicable to $\mathcal{C}_1, \mathcal{C}_2$. The map $\mathbf{ML}_2 \to \mathbf{ML}_2$ replacing each occurrence of $\Box_i$ with $\Box_i^{l_i}$ is a reduction from $\mathrm{Log}(\mathcal{C}_1 \times \mathcal{C}_2)$ to $\mathbf{K}(\Gamma_1) \times \mathbf{K}(\Gamma_2)$, hence the undecidability. By Lemma 4.3, $\mathbf{K}(\Gamma_1) \times \mathbf{K}(\Gamma_2)$ is finitely axiomatizable, hence the lack of FMP. $\Box$

6 Products with the FMP

We outline a proof of Theorem 5.1(2) similar to the reasoning in [8].

Definition 6.1 *A relation $E \subseteq W_1 \times W_2$ is a* temporal bisimulation *between the frames (W_1, R_1) and (W_2, R_2) if $R_2 \circ E = E \circ R_1$ and $R_1 \circ E^{-1} = E^{-1} \circ R_2$. Two relations S_1 and S_2 on the same set* strongly commute *if $S_1 \circ S_2 = S_2 \circ S_1$ and $S_1 \circ S_2^{-1} = S_2^{-1} \circ S_1$. A frame (W, R)* admits temporal bisimulation *if for any finite $W/\!\sim$ there exists an equivalence relation of finite index $\approx$ strongly commuting with R such that $\approx \subseteq \sim$.*

Lemma 6.2 *Let Γ be a tree-theory and E a temporal bisimulation between $\mathfrak{F}$ and $\mathfrak{G}$; then E is also a temporal bisimulation between $\mathfrak{F}^\Gamma$ and $\mathfrak{G}^\Gamma$.*

Lemma 6.3 ("filtration via bisimulation") *Let Γ_1, Γ_2 be tree-theories and (W, R_1, R_2) a 2-frame such that:*

(1) $(W, R_i) \models \Gamma_i$ and $\mathbf{K}(\Gamma_i)$ admits filtration with respect to (W, R_i), for $i \in \{1, 2\}$;

(2) R_1 and R_2 strongly commute; and

(3) (W, R_1) admits temporal bisimulation.

Then $[\mathbf{K}(\Gamma_1); \mathbf{K}(\Gamma_2)]$ admits filtration with respect to (W, R_1, R_2).

Proof outline Fix a valuation $\mathfrak{V}$ on W and a finite subformula-closed set $\Sigma \subseteq \mathbf{ML}_2$. It may be shown (using (1) only) that there exists an equivalence relation $\sim$ such that $(W/\!\sim, (R_1^\sim)^{\Gamma_1}, (R_2^\sim)^{\Gamma_2}, \mathfrak{V}^\sim)$ is a finite Σ-filtration of $(W, R_1, R_2, \mathfrak{V})$.

Let $\approx$ be the union of all relations containing in $\sim$ and strongly commuting with R_1. Since (W, R_1) admits temporal bisimulation, $\approx$ is an equivalence

[2] Decidability does not necessarily follow, as some uniform Horn logics are not finitely axiomatizable.

relation of finite index. We have $\approx \circ R_1 = R_1 \circ \approx$; thus $R_1^{\approx}$ and $R_2^{\approx}$ strongly commute; hence $(R_1^{\approx})^{\Gamma_1}$ and $(R_2^{\approx})^{\Gamma_2}$ strongly commute by Lemma 6.2; therefore $(W/{\approx}, (R_1^{\approx})^{\Gamma_1}, (R_2^{\approx})^{\Gamma_2})$ is a frame for $[\mathbf{K}(\Gamma_1); \mathbf{K}(\Gamma_2)]$. $\square$

Definition 6.4 A frame is a *pseudo-tree of height* 1 if it is pseudo-finite. A frame (W, R) is a *pseudo-tree of height* $h > 1$ if W can be represented in the form $W_0 \cup \bigsqcup_{j \in J} W_j$ such that:
(1) $|W_0 \cap W_j| = 1$ for each $j \in J$;
(2) $R = R|_{W_0} \cup \bigsqcup_{j \in J} R|_{W_j}$;
(3) $(W_0, R|_{W_0})$ is a pseudo-tree of height $h - 1$; and
(4) for some s, all $(W_j, R|_{W_j})$ are s-pseudo-finite.

Lemma 6.5 *Every pseudo-tree admits temporal bisimulation.*

Proof outline By induction on height, in the same way as for trees of finite height in [8]. $\square$

Lemma 6.6 *Let Γ be a strong tree-theory and $\mathfrak{T}$ a tree. Then $\mathfrak{T}^{\Gamma}$ coincides with the Horn Γ-closure of a pseudo-tree.*

Proof Follows from Lemma 4.2. $\square$

Definition 6.7 The d-truncation of a pointed frame (W, R, w) is the frame $(W_d, R|_{W_d}, w)$, where $W_d := R^{\le d}(w)$. Pointed frames (W, R, w) and (V, S, v), are *d-indistinguishable* if their d-truncations are isomorphic.

Lemma 6.8 *Let Γ be a reflexive-symmetric or uniform tree-theory, $d > 0$. Then every tree $\mathfrak{T}$ with root w has a subtree $\mathfrak{T}'$ of finite height such that $(\mathfrak{T}^{\Gamma}, w)$ and $(\mathfrak{T}'^{\Gamma}, w)$ are d-indistinguishable.*

Proof of Theorem 5.1(2) Set $L := \mathbf{K}(\Gamma_1) \times \mathbf{K}(\Gamma_2)$. Consider $\varphi \notin L$; let d be its modal depth. Choose trees $\mathfrak{T}_1, \mathfrak{T}_2$ as in Lemma 3.5. Now for each $i \in \{1, 2\}$, depending on the type of Γ_i, apply one of Lemmas 4.4, 6.6, or 6.8. It follows that there exist frames $\mathfrak{F}_i$ d-indistinguishable from (or coinciding with) $\mathfrak{T}_i^{\Gamma_i}$ such that Lemma 6.3 is applicable to $\mathfrak{F}_1 \times \mathfrak{F}_2$. $\square$

7 Adding Variable-Free Axioms

Theorem 7.1 *Let Γ_1, Γ_2 be reflexive-symmetric, transitive, or strong tree-theories, and λ_1, λ_2 variable-free formulas.*

1) If both Γ_1 and Γ_2 are transitive, and $\mathbf{K}(\Gamma_i) + \lambda_i \nvdash \Box^n \bot$ for every $i \in \{1, 2\}$ and $n > 0$, then $(\mathbf{K}(\Gamma_1) + \lambda_1) \times (\mathbf{K}(\Gamma_2) + \lambda_2)$ is undecidable and lacks the fmp.

2) In all other cases, $(\mathbf{K}(\Gamma_1) + \lambda_1) \times (\mathbf{K}(\Gamma_2) + \lambda_2)$ is decidable and has the fmp.

Note that Theorem 7.1 does not cover *uniform* tree-theories; that case remains unclear. The proof is the same as for Theorem 5.1, except that we use the following lemma instead of Lemma 6.8.

Lemma 7.2 *Let Γ be a reflexive-symmetric tree-theory, λ a variable-free formula, $\mathfrak{T} = (W, R)$ a tree with root w such that $\mathfrak{T}^{\Gamma} \models \lambda$, and $d > 0$ an integer. Then there exists a pseudo-tree $\mathfrak{F}$ with a node v such that*

(1) $\mathfrak{F}^\Gamma \models \lambda$, *and*
(2) $(\mathfrak{T}^\Gamma, w)$ *and* $(\mathfrak{F}^\Gamma, v)$ *are d-indistinguishable.*

Proof outline By Lemma 4.3, we can assume that Γ is finite. One can derive that, for some finite $W/{\sim}$, the minimal filtration $(W/{\sim}, (R^\Gamma)^{\sim})$ is a frame for both Γ and λ.

Set $x \approx y$ if (1) $x \sim y$ and (2) either $x = y$ or the least common ancestor of x and y is not in $R^{\leq d}(w)$. Note that $(W/{\approx}, R^{\approx})$ is a pseudo-tree and that $(R^{\approx})^\Gamma = (R^\Gamma)^{\approx}$. One can show that (W, R^Γ, w) and $(W/{\approx}, (R^\Gamma)^{\approx}, [w])$ are d-indistinguishable. □

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