Рекурсивная неотделимость в модальных и суперинтуиционистских предикатных логиках

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Motivation: classical decision problem

- Classical decision problem (David Hilbert): find an algorithm deciding validity in the classical first-order logic **QCl**.
- Solution (Alonzo Church, Alan Turing 1936/1937): **QCl** is undecidable.
- Classical decision problem as a classification problem: identify the "maximal" decidable and the "minimal" undecidable fragments of **QCl**.
- Criteria:
  - the quantifier prefix: $\exists^*\forall^*$ decidable, $\forall^3\exists^*$ undecidable;
  - the number of variables: 2 decidable, 3 undecidable;
  - the number and arity of predicate letters: any number of monadic decidable, a single binary undecidable.

Motivation: non-classical logics

- Decision problem for non-classical logics as a classification problem: identify the "maximal" decidable and the "minimal" undecidable fragments of modal and superintuitionistic predicate logics.
- S. Kripke (1962): Every modal predicate logic validated by **QS5**-frames is undecidable with two monadic predicate letters: write $\Diamond(P_1(x) \wedge P_2(y))$ for $Q(x,y)$ to obtain an embedding of an undecidable fragment of **QCl** ("Kripke trick").
- Single-variable fragments are, as a rule, decidable (K. Segerberg, G. Fisher Servi, H. Ono, G. Mints).
- F. Wolter and M. Zakharyaschev (2001): Monodic fragments of a number of logics are decidable.

Motivation: non-classical logics

- S. Maslov, G. Mints, V. Orevkov (1965):
  The intuitionistic predicate logic **QInt** is undecidable with a single monadic predicate letter.
- D. Gabbay, V. Shehtman (1993):
  Most natural modal and superintuitionistic predicate logics with the constant domain axiom are undecidable in languages with two individual variables.
- R. Konchakov, A. Kurucz, M. Zakharyaschev (2005):
  QInt and every modal logic validated by **QS5**-frames are undecidable with two individual variables.
- M. Rybakov, D. Shkatov (2018):
  QInt, **QK**, **QT**, **QS4**, as well as a number of related logics, are undecidable in languages with two individual variables and a single monadic predicate letter.

Questions

- E. Zolin (2019): Are the monadic fragments of modal predicate logics defined by classes of finite Kripke frames decidable in languages with finitely many individual variables?
  - Partial answer:
    M. Rybakov, D. Shkatov (2021): $\mathbf{QInt}_{\mathit{wfin}}$, $\mathbf{QK}_{\mathit{wfin}}$, $\mathbf{QT}_{\mathit{wfin}}$, $\mathbf{QS4}_{\mathit{wfin}}$, as well as a number of related logics, are undecidable in languages with three individual variables and a single monadic predicate letter.
- V. Shehtman (2023): Are the monadic fragments of a modal predicate logic and the monadic fragment of the logic of its finite Kripke frames recursively separable (distinguishable)?

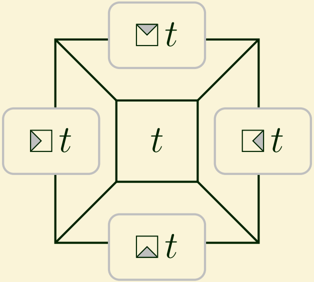
This talk

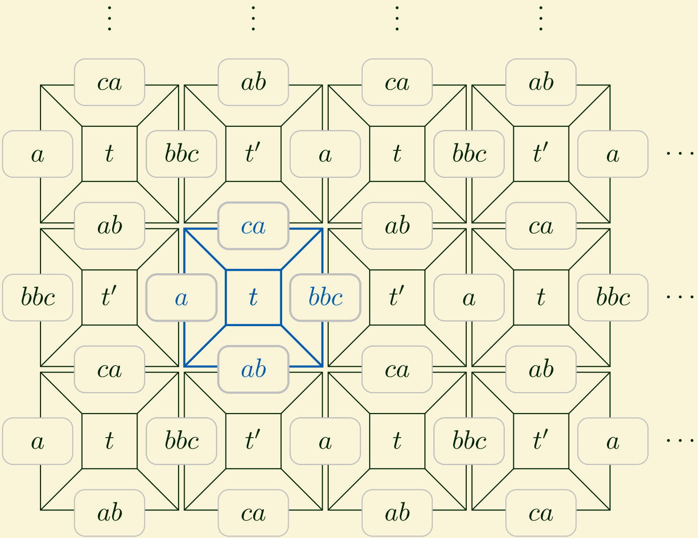
Let T_{fin} be the theory of finite T-models.
Let L_{wfin} be the logic of finite (by the number of worlds) L-frames.

In this talk, we discuss the following:

- Theories (between) **QCl** and theory $\mathbf{SIB}_{fin}$ are not recursively separable in the language containing a single binary predicate letter and three individual variables.
- Logics (between) **QK** and $\mathbf{QS5}_{wfin}$ are not recursively separable in the language containing a single unary predicate letter and three individual variables.
- Logics (between) **QK** and $\mathbf{QGL}_{wfin} \oplus \boldsymbol{bf}$ are not recursively separable in the language containing a single unary predicate letter and two individual variables.
- Logics (between) **QInt** and $\mathbf{QKC}_{wfin} + \boldsymbol{cd}$ are not recursively separable in the language containing a single unary predicate letter and two individual variables.

Tile and tile type

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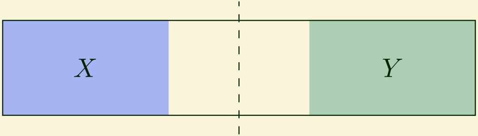
Recursively inseparable sets and logics

Let X and Y be subsets of $\mathbb{N}$ such that $X \cap Y = \varnothing$. Then X and Y are called *recursively separable* if there exists a recursive subset Z of $\mathbb{N}$ such that $X \subseteq Z$ and $Y \cap Z = \varnothing$; if there is no such Z, then X and Y are called *recursively inseparable*.



Let X and Y be subsets of $\mathbb{N}$ such that $X \subseteq Y$. Then X and Y are called *recursively distinguishable* if there exists a recursive subset Z of $\mathbb{N}$ such that $X \subseteq Z \subseteq Y$; if there is no such Z, then X and Y are called *recursively indistinguishable*.

Notice that

$$\begin{array}{lcl} X \text{ and } Y \text{ are rec. sep.} & \Longleftrightarrow & X \text{ and } \mathbb{N} \setminus Y \text{ are rec. dist.;} \\ X \text{ and } Y \text{ are rec. dist.} & \Longleftrightarrow & X \text{ and } \mathbb{N} \setminus Y \text{ are rec. sep.} \end{array}$$

Step 1: Fix a Turing machine

Let $\mathbb{X}$ and $\mathbb{Y}$ be recursively enumerable recursively inseparable sets. Then there is a partial recursive function $f_{\mathbb{X}\mathbb{Y}} \colon \mathbb{N} \to \mathbb{N}$ distinguishing $\mathbb{X}$ and $\mathbb{Y}$:

$$f_{\mathbb{X}\mathbb{Y}}(x) \;=\; \begin{cases} 0 & \text{if } x \in \mathbb{X}; \\ 1 & \text{if } x \in \mathbb{Y}; \\ \text{undefined} & \text{if } x \notin \mathbb{X} \cup \mathbb{Y}. \end{cases}$$

It is computable by a Turing machine $M_0 = \langle \Sigma_0, Q_0, q_0, F_0, \delta_0 \rangle$.
We may assume that M_0 has two halting states $q_{\mathbb{X}}$ and $q_{\mathbb{Y}}$ such that, for every $m \in \mathbb{N}$,

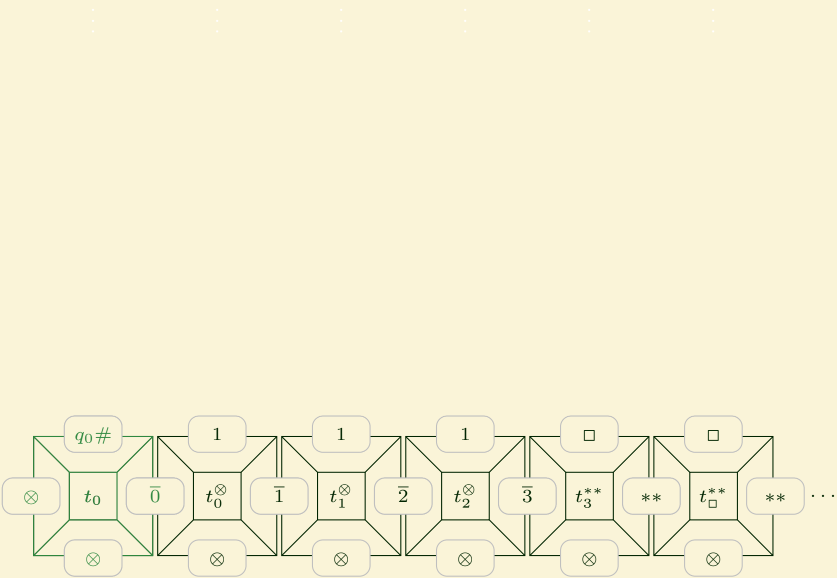
- if $m \in \mathbb{X}$, then $q_{\mathbb{X}}!M_0(\overline{m})$;
- if $m \in \mathbb{Y}$, then $q_{\mathbb{Y}}!M_0(\overline{m})$;

also, notice that

- if $m \notin \mathbb{X} \cup \mathbb{Y}$, then $\neg!M_0(\overline{m})$.

We add the following technical commands to M_0:
$q_{\mathbb{X}}\# \to q_{\mathbb{X}}\#S$ and $q_{\mathbb{Y}}\# \to q_{\mathbb{Y}}\#S$.

Step 2: Fix a set T_n of tile types, for every $n \in \mathbb{N}$



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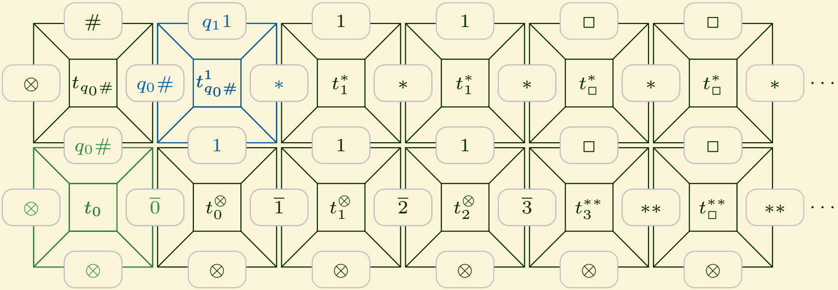

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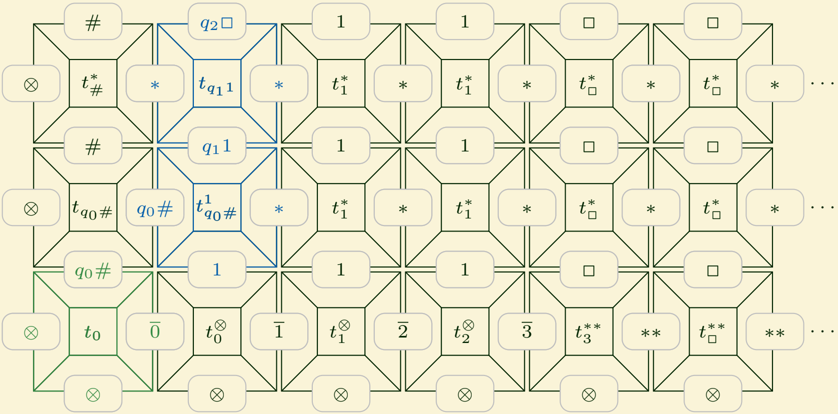

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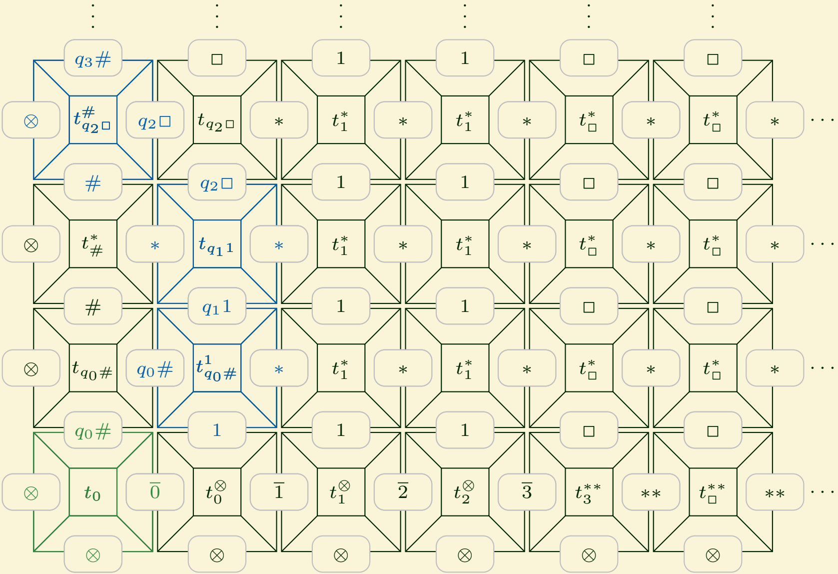
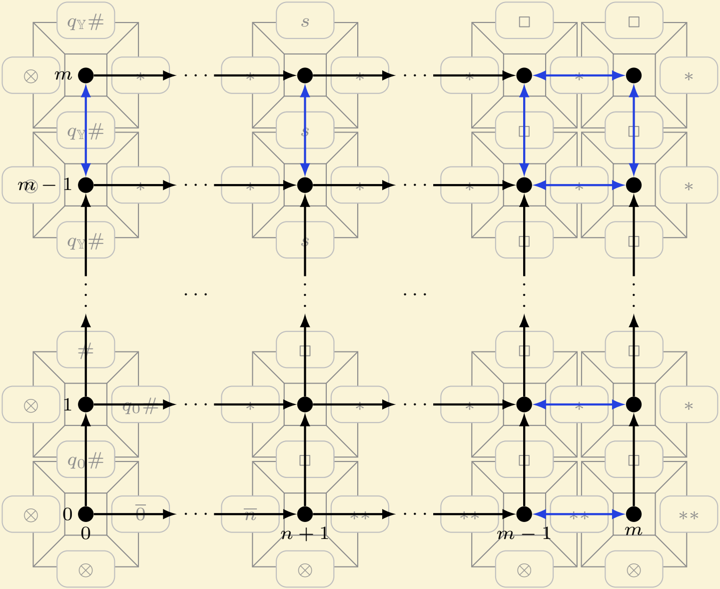


Finite model for a special T_n-tiling



Step 3: Writing formulas

$$H_n(x,y) \;=\; P(x,y) \wedge \bigvee_{i,j=0}^{k_n} \{P_i(x) \wedge P_j(y) : \boxed{\lhd}\, t_i^n = \boxed{\rhd}\, t_j^n\};$$

$$V_n(x,y) \;=\; P(x,y) \wedge \bigvee_{i,j=0}^{k_n} \{P_i(x) \wedge P_j(y) : \boxed{\triangledown}\, t_i^n = \boxed{\vartriangle}\, t_j^n\};$$

$$TC_0 \;=\; \forall x \bigvee_{i=0}^{k_n} \Big(P_i(x) \wedge \bigwedge_{j \neq i} \neg P_j(x)\Big);$$

$$TC_1 \;=\; \forall x \exists y\, H_n(x,y);$$

$$TC_2 \;=\; \forall x \exists y\, V_n(x,y);$$

$$TC_3 \;=\; \forall x \forall y\, (\exists z\, (H_n(x,z) \wedge V_n(z,y)) \leftrightarrow \exists z\, (V_n(x,z) \wedge H_n(z,y)));$$

$$TC_4 \;=\; \exists x\, P_0(x);$$

$$\mathit{Tiling}_n \;=\; TC_0 \wedge TC_1 \wedge TC_2 \wedge TC_3 \wedge TC_4;$$

$$\mathit{Tiling}_n^{\mathbb{X}} \;=\; \mathit{Tiling}_n \to \exists x\, P_1(x); \qquad \longleftarrow \quad \text{“}n \in \mathbb{X}\text{”} \qquad t_1 = t_{q_{\mathbb{X}}\#}$$

$$\mathit{Tiling}_n^{\mathbb{Y}} \;=\; \mathit{Tiling}_n \to \exists x\, P_2(x). \qquad \longleftarrow \quad \text{“}n \in \mathbb{Y}\text{”} \qquad t_2 = t_{q_{\mathbb{Y}}\#}$$

Statements

Lemma

If $n \in \mathbb{X}$, then $Tiling_n^{\mathbb{X}} \in \mathbf{QCl}$.

Lemma

If $n \in \mathbb{Y}$, then $Tiling_n^{\mathbb{X}} \notin \mathbf{QCl}_{fin}$.

Theorem (Trakhtenbrot)

Logics $\mathbf{QCl}$ and $\mathbf{QCl}_{fin}$ are recursively indistinguishable in a language containing a binary predicate letter, an infinite supply of unary predicate letters, and three individual variables.

Step 4: Relativization

Let G be a new unary predicate letter; define $\forall_G$ and $\exists_G$ by

$$\begin{array}{rcl} \forall_G x\,\varphi & = & \forall x\,(G(x) \to \varphi); \\ \exists_G x\,\varphi & = & \exists x\,(G(x) \wedge \varphi). \end{array}$$

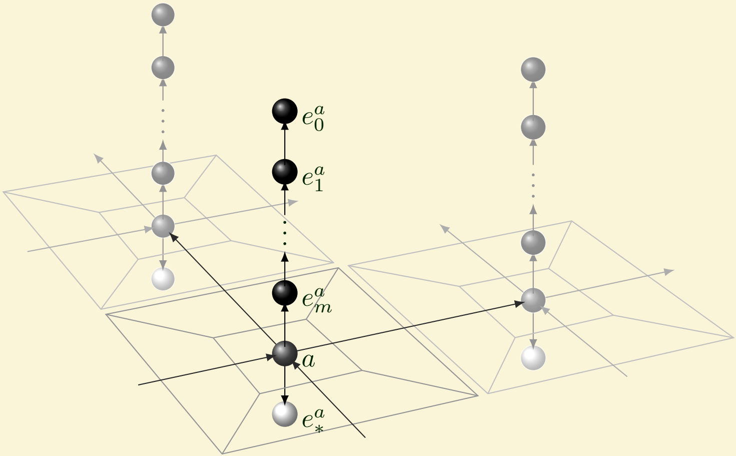
For a formula φ, denote by φ_G the formula obtained from φ by replacing each quantifier $\forall x$ or $\exists x$ with $\forall_G x$ or $\exists_G x$, respectively.

Lemma

Let φ be a closed formula without occurrences of G. Then

$$\begin{array}{lcl} \varphi \in \mathbf{QCl} & \iff & \exists x\,G(x) \to \varphi_G \in \mathbf{QCl}. \\ \varphi \in \mathbf{QCl}_{\mathit{fin}} & \iff & \exists x\,G(x) \to \varphi_G \in \mathbf{QCl}_{\mathit{fin}}. \end{array}$$

Step 5: Eliminating unary predicate letters



Writing formulas

Let us define formulas we shall use to simulate unary predicate letters:

$$
\begin{array}{rcl}
\varepsilon^y(x) & = & \neg\exists y\, P(x,y);\\
\tau^y_0(x) & = & \exists y\,(\neg G(y)\wedge P(x,y)\wedge \varepsilon^x(y));\\
\tau^y_{k+1}(x) & = & \exists y\,(\neg G(y)\wedge P(x,y)\wedge \tau^x_k(y));\\
\mathit{tile}_k(x) & = & \tau^y_k(x); \qquad \mathit{tile}_k(y) \;=\; \tau^x_k(y); \qquad \mathit{tile}_k(z) \;=\; \tau^x_k(z);\\
\gamma^y(x) & = & \neg P(x,x)\wedge\exists y\,(P(x,y)\wedge P(y,y));\\
\mathit{grid}(x) & = & \gamma^y(x); \qquad \mathit{grid}(y) \;=\; \gamma^x(y); \qquad \mathit{grid}(z) \;=\; \gamma^x(z).
\end{array}
$$

Define $S_0\,\mathit{Tiling}^{\mathbb{X}}_n$ to be obtained from $\exists x\, G(x) \to (\mathit{Tiling}^{\mathbb{X}}_n)_G$ by replacing

- $P_k(x)$, $P_k(y)$, $P_k(z)$ with $\mathit{tile}_k(x)$, $\mathit{tile}_k(y)$, $\mathit{tile}_k(z)$;
- $G(x)$, $G(y)$, $G(z)$ with $\mathit{grid}(x)$, $\mathit{grid}(y)$, $\mathit{grid}(z)$,

respectively.

Statements

Lemma

If $n \in \mathbb{X}$, *then* $S_0\mathit{Tiling}_n^{\mathbb{X}} \in \mathbf{QCl}$.

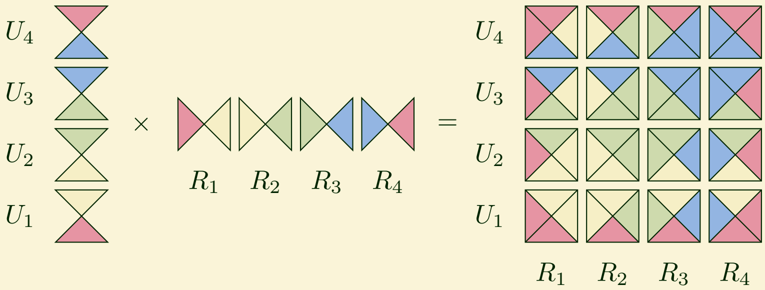
Lemma

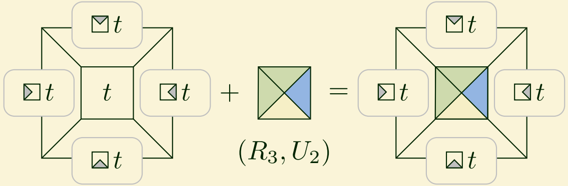
If $n \in \mathbb{Y}$, *then* $S_0\mathit{Tiling}_n^{\mathbb{X}} \notin \mathbf{QCl}_{\mathit{fin}}$.

Theorem

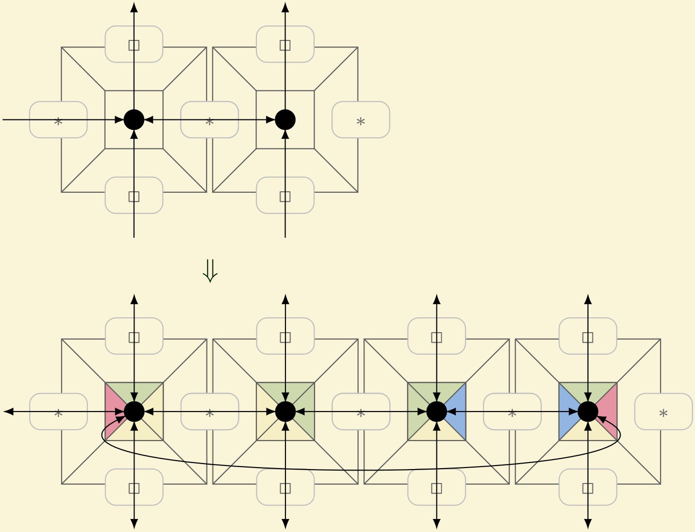
Logics $\mathbf{QCl}$ *and* $\mathbf{QCl}_{\mathit{fin}}$ *are recursively indistinguishable in a language containing a binary predicate letter and three individual variables.*

Step 6: Adding inner tiles

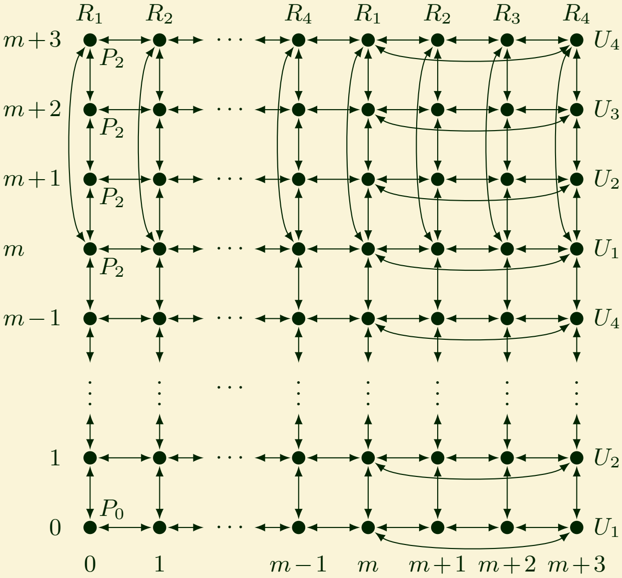




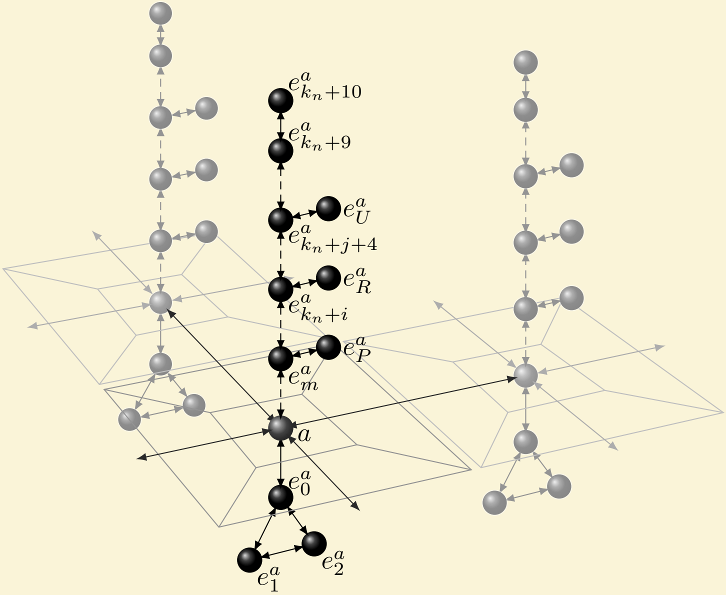
Directed cycles in sib-models



Finite model for a T_n-tiling



Step 7: Eliminating unary letters using a sib-relation



Statements

Theorem

Theories $\mathbf{QCl}$ *and* $\mathbf{SIB}_{fin}$ *are recursively indistinguishable in a language with a binary predicate letter and three individual variables.*

Corollary

Theories $\mathbf{QCl}$ *and* $\mathbf{SRB}_{fin}$ *are recursively indistinguishable in a language with a binary predicate letter and three individual variables.*

Corollary

Theories $\mathbf{SIB}$ *and* $\mathbf{SIB}_{fin}$*, as well as* $\mathbf{SRB}$ *and* $\mathbf{SRB}_{fin}$ *are recursively indistinguishable in a language containing a binary predicate letter and three individual variables.*

Theorem

Let Γ *and* Γ' *be theories of a binary predicate such that* $\mathbf{QCl}^{bin} \subseteq \Gamma \subseteq \Gamma'$ *and also* $\Gamma' \subseteq \mathbf{SIB}$ *or* $\Gamma' \subseteq \mathbf{SRB}$*. Then* Γ *and* Γ'_{fin} *are recursively indistinguishable in a language with three variables.*

Modal and intuitionistic predicate languages

Intuitionistic formulas:

$$\varphi \ ::= \ P^n(x_1,\ldots,x_n) \mid \bot \mid (\varphi\wedge\varphi) \mid (\varphi\vee\varphi) \mid (\varphi\to\varphi) \mid \forall x\,\varphi \mid \exists x\,\varphi$$

Modal formulas:

$$\varphi \ ::= \ P^n(x_1,\ldots,x_n) \mid \bot \mid (\varphi\wedge\varphi) \mid (\varphi\vee\varphi) \mid (\varphi\to\varphi) \mid \forall x\,\varphi \mid \exists x\,\varphi \mid \Box\varphi$$

Standard abbreviations:

$$\begin{array}{lcl}\neg\varphi & = & \varphi\to\bot;\\ \varphi\leftrightarrow\psi & = & (\varphi\to\psi)\wedge(\psi\to\varphi);\\ \Diamond\varphi & = & \neg\Box\neg\varphi.\end{array}$$

Kripke semantics

Kripke frame is a pair $\mathfrak{F} = \langle W, R\rangle$; for the intuitionistic language R is reflexive, transitive, and antisymmetric.

Expanding domains. For a frame $\mathfrak{F} = \langle W, R\rangle$ consider a system $(D_w)_{w\in W}$ of non-empty sets (domains) such that

$$(*) \quad wRw' \implies D_w \subseteq D_{w'}.$$

For every $w \in W$ define a classical model $M_w = (D_w, I_w)$.
For the intuitionistic case we additionally claim:

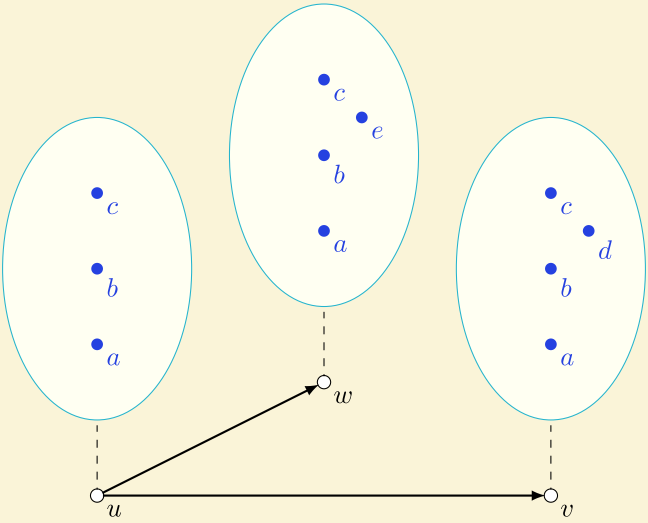
$$wRw' \implies I_w(P^n) \subseteq I_{w'}(P^n).$$

Kripke model is a tuple $\mathfrak{M} = (W, R, D, I)$, where $D = (D_w)_{w\in W}$ and $I = (I_w)_{w\in W}$.

Locally constant domains. Replace $(*)$ with cd-condition:

$$(**) \quad wRw' \implies D_w = D_{w'}.$$

Predicate Kripke frames: an example



Kripke semantics

Truth relation (intuitionistic language):

- $\mathfrak{M}, w \models^g P(x_1, \ldots, x_n)$ if $\langle g(x_1), \ldots, g(x_n) \rangle \in P^w$;
- $\mathfrak{M}, w \not\models^g \bot$;
- $\mathfrak{M}, w \models^g \varphi \wedge \psi$ if $\mathfrak{M}, w \models^g \varphi$ and $\mathfrak{M}, w \models^g \psi$;
- $\mathfrak{M}, w \models^g \varphi \vee \psi$ if $\mathfrak{M}, w \models^g \varphi$ or $\mathfrak{M}, w \models^g \psi$;
- $\mathfrak{M}, w \models^g \varphi \to \psi$ if $\mathfrak{M}, w' \models^g \varphi$ implies $\mathfrak{M}, w' \models^g \psi$, for any $w' \in R(w)$;
- $\mathfrak{M}, w \models^g \exists x\, \varphi$ if $\mathfrak{M}, w \models^{g'} \varphi$, for some g' s.t. $g' \stackrel{x}{=} g$ and $g'(x) \in D_w$;
- $\mathfrak{M}, w \models^g \forall x\, \varphi$ if $\mathfrak{M}, w' \models^{g'} \varphi$, for every $w' \in R(w)$ and every g' s.t. $g' \stackrel{x}{=} g$ and $g'(x) \in D_{w'}$.

Truth relation (modal language):

- $\mathfrak{M}, w \models^g P(x_1, \ldots, x_n)$ if $\langle g(x_1), \ldots, g(x_n) \rangle \in P^w$;
- $\mathfrak{M}, w \not\models^g \bot$;
- $\mathfrak{M}, w \models^g \varphi \wedge \psi$ if $\mathfrak{M}, w \models^g \varphi$ and $\mathfrak{M}, w \models^g \psi$;
- $\mathfrak{M}, w \models^g \varphi \vee \psi$ if $\mathfrak{M}, w \models^g \varphi$ or $\mathfrak{M}, w \models^g \psi$;
- $\mathfrak{M}, w \models^g \varphi \to \psi$ if $\mathfrak{M}, w \models^g \varphi$ implies $\mathfrak{M}, w \models^g \psi$;
- $\mathfrak{M}, w \models^g \exists x\, \varphi$ if $\mathfrak{M}, w \models^{g'} \varphi$, for some g' s.t. $g' \stackrel{x}{=} g$ and $g'(x) \in D_w$;
- $\mathfrak{M}, w \models^g \forall x\, \varphi$ if $\mathfrak{M}, w \models^{g'} \varphi$, for every g' s.t. $g' \stackrel{x}{=} g$ and $g'(x) \in D_w$;
- $\mathfrak{M}, w \models^g \Box\varphi$ if $\mathfrak{M}, w' \models^g \varphi$, for every $w' \in R(w)$.

- $\mathfrak{M}, w \models \varphi(x_1, \ldots, x_n)$ if $\mathfrak{M}, w \models^g \varphi(x_1, \ldots, x_n)$, for every g such that $g(x_1), \ldots, g(x_n) \in D_w$;
- $\mathfrak{M} \models \varphi$ if $\mathfrak{M}, w \models \varphi$, for every $w \in W$;
- $\mathfrak{F} \models \varphi$ if $\mathfrak{M} \models \varphi$, for every model $\mathfrak{M}$ based over $\mathfrak{F}$.

Logics

The logics under consideration are:

- **QK**, the modal logic of all Kripke frames;
- $\mathbf{Q}L = \mathbf{QK} \oplus L$, for a normal modal propositional logic L;
- **QInt**, the logic of all intuitionistic Kripke frames;
- **QKC**, the logic of convergent intuitionistic Kripke frames;
- L_{wfin}, the logic of all finite Kripke frames of L.

The Barcan formula: $\boldsymbol{bf} = \forall x\, \Box P(x) \to \Box \forall x\, P(x)$.
It is valid on a frame with domains if, and only if, the frame satisfies the locally constant domain condition.

KHC-classes of frames

A Kripke frame $\mathfrak{F} = \langle W, R \rangle$ satisfies the *Kripke–Hughes–Cresswell condition*, or, for short, *KHC*, if $R(w)$ is infinite, for some $w \in W$.
A logic L is *KHC-friendly* if there exists an L-frame satisfying KHC.

- S. Kripke (1962), G. Hughes and M. Cresswell (1996):
  Monadic fragments of KHC-friendly modal predicate logics are undecidable.

Let $\mathscr{C}$ be a class of Kripke frames; we say that

- $\mathscr{C}$ satisfies the *weak Kripke–Hughes–Cresswell condition*, or, for short, *wKHC*, if, for every $n \in \mathbb{N}$, there exists a Kripke frame $\langle W, R \rangle \in \mathscr{C}$ with $w \in W$ such that $|R(w)| \geqslant n$.
- $\mathscr{C}$ is a *Skvortsov class* if the class of finite frames from $\mathscr{C}$ satisfies wKHC.

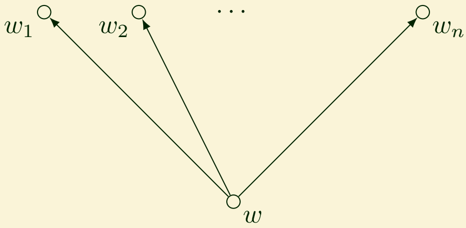
Simulating sib-relation by a monadic predicate

Let Q be a unary predicate letter.
Let S_2 be a function substituting $\Box(Q(x_1) \vee Q(x_2))$ for $P(x_1, x_2)$.

Lemma

Let L be a modal predicate logic such that $\mathbf{QCl} \subseteq L$ and the class of L-frames is a Skvortsov class. Then, for every classical formula φ, containing no predicate letters except the binary letter P,

$$\varphi \notin \mathbf{SIB}_{\mathit{fin}} \implies S_2\varphi \notin L_{\mathit{wfin}} \oplus \boldsymbol{bf}.$$



Simulating sib-relation by a monadic predicate

Theorem

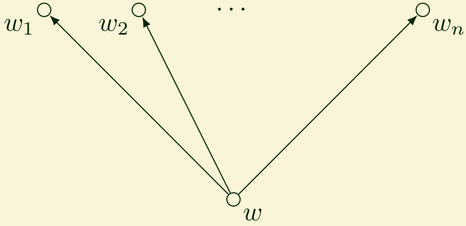
Let L and L' be modal predicate logics such that $\mathbf{QCl} \subseteq L \subseteq L'$ *and the class of L'-frames is a Skvortsov class. Then L and* $L'_{\mathit{wfin}} \oplus \boldsymbol{bf}$ *are recursively indistinguishable in the language with a single unary predicate letter and three variables.*

Corollary

Let L be one of **QK**, **QT**, **QD**, **QKB**, **QKTB**, **QK4**, **QS4**, **QK5**, **QS5**, **QK45**, **QKD45**, **QK4B**, **QGL**, **QGrz**, $\mathbf{QK4} \oplus \boldsymbol{bd}_n$, $\mathbf{QS4} \oplus \boldsymbol{bd}_n$, $\mathbf{QK4} \oplus \boldsymbol{bw}_m$, $\mathbf{QS4} \oplus \boldsymbol{bw}_m$, *where $n \geqslant 2$ and $m \geqslant 1$. Then L and L_{wfin} are recursively indistinguishable in the language with a single unary predicate letter and three individual variables.*

Remark on superintuitionistic predicate logics

Let $\boldsymbol{cd} = \forall x\,(S(x) \vee q) \to (\forall x\,S(x) \vee q)$.
Let Q be a unary predicate letter.
Use the function substiting $(Q(x_1) \wedge Q(x_2) \to p) \vee q$ for $P(x_1, x_2)$.



Theorem

Let L be a logic between **QInt** *and* $\mathbf{QKC} + \boldsymbol{cd}$*. Then the positive fragments of L and L_{wfin} are recursively indistinguishable in the language with a single unary predicate letter and three individual variables.*

Monadic fragments with two variables

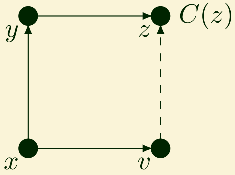
$$
\begin{array}{rcl}
TC_1^{\Box} & = & \forall x \exists y\, H_n(x,y); \qquad = TC_1 \\
TC_2^{\Box} & = & \forall x \exists y\, V_n(x,y); \qquad = TC_2 \\
TC_3 & = & \forall x \forall y\, (\exists z\, (H_n(x,z) \wedge V_n(z,y)) \leftrightarrow \exists z\, (V_n(x,z) \wedge H_n(z,y)); \\
TC_3^{\Box} & = & \Box \forall x \forall y\, (V_n(x,y) \wedge \exists x\, (C(x) \wedge H_n(y,x)) \to \\
 & & \qquad \forall y\, (H_n(x,y) \to \forall x\, (C(x) \to V_n(y,x)))); \\
TC_4^{\Box} & = & \exists x\, P_0(x); \qquad = TC_4 \\
TC_5^{\Box} & = & \forall x\, \Diamond C(x); \\
TC_6^{\Box} & = & \forall x \forall y\, (V_n(x,y) \to \Box V_n(x,y)); \\
TC_7^{\Box} & = & \forall x \forall y\, (H_n(x,y) \to \Box H_n(x,y)); \\
TC_8^{\Box} & = & \forall x \forall y\, (\Diamond V_n(x,y) \to V_n(x,y)); \\
TC_9^{\Box} & = & \forall x \forall y\, (\Diamond H_n(x,y) \to H_n(x,y)); \\
\mathit{Tiling}_n^{\Box} & = & \bigwedge_{i=0}^{9} TC_i^{\Box}; \\
M^{\Box}\,\mathit{Tiling}_n^{\mathbb{X}} & = & \mathit{Tiling}_n^{\Box} \to \exists x\, P_1(x).
\end{array}
$$

Monadic fragments with two variables

$$TC_3^{\Box} \;=\; \Box\forall x\forall y\,(V_n(x,y)\wedge\exists x\,(C(x)\wedge H_n(y,x))\to \forall y\,(H_n(x,y)\to\forall x\,(C(x)\to V_n(y,x)))).$$

Let us rewrite it:

$$TC_3^{\Box} \;=\; \Box\forall x\forall y\,(V_n(x,y)\wedge\exists z\,(C(z)\wedge H_n(y,z))\to \forall v\,(H_n(x,v)\to\forall z\,(C(z)\to V_n(v,z)))).$$



Statements

Lemma

If $n \in \mathbb{X}$, then $M^{\Box}\mathit{Tiling}_n^{\mathbb{X}} \in \mathbf{QK}$.

Lemma

If $n \in \mathbb{Y}$, then $M^{\Box}\mathit{Tiling}_n^{\mathbb{X}} \notin \mathbf{QGL}_{\mathit{wfin}}, \mathbf{QGrz}_{\mathit{wfin}}, \mathbf{QS5}_{\mathit{wfin}}$.

Some further manipulations with languages give us the following.

Theorem

Let L be a logic between $\mathbf{QK}$ and $\mathbf{QGL}$ or between $\mathbf{QK}$ and $\mathbf{QGrz}$. Then L and L_{wfin} are recursively indistinguishable in the language with a single unary predicate letter and two individual variables.

Theorem

Let L be a logic between $\mathbf{QInt}$ and $\mathbf{QKC} + \boldsymbol{cd}$. Then the positive fragments of L and L_{wfin} are recursively indistinguishable in the language with a single unary predicate letter and two individual variables.

Some clopen :) questions

Question

Is **QS5** with two individual variables decidable in languages with finitely many unary predicate letters?

Theorem

It is undecidable with three unary predicate letters.

Conjecture

It is undecidable with two unary predicate letters. Yes

Conjecture

It is decidable with a single unary predicate letter. No

Question

Is the superintuitionistic predicate logic of linear Kripke frames decidable in languages with two individual variables? No

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Thanks

Thank you!